Lecture 13
Foundations of Statics

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Mechanics of Manipulation

Today's outline

Preview of statics

- We will adopt Newton's hypothesis that particles interact through forces.
- We can then show that rigid bodies interact through *wrenches*.
- Screw theory applies to wrenches.
- Wrenches and twists are dual.
- We also get:
  - Line of force;
  - Screw coordinates applied to statics;
  - Reciprocal product of twist and wrench;
  - Zero Moment Point (ZMP), and its generalization.

What is force?

- You cannot measure force, only its effects: deformation of structures, acceleration.
- We could start from Newton's laws, but instead we hypothesize:
  - A force applied to a particle is a vector.
  - The motion of a particle is determined by the vector sum of all applied forces.
  - A particle remains at rest only if that vector sum is zero.

Moment of force about a line

Definition

- Let l be line through origin with direction $\hat{\mathbf{l}}$,
- Let $\mathbf{f}$ act at $\mathbf{x}$.
- Then the moment of force (or the torque) of f about l is given by:

$$n_l = \hat{\mathbf{l}} \cdot (\mathbf{x} \times \mathbf{f})$$

Moment of force about a point

Definition

- Let l be line through origin with direction $\hat{\mathbf{l}}$,
- Let $\mathbf{f}$ act at $\mathbf{x}$.
- Then the moment of force (or the torque) of f about O is given by:

$$\mathbf{n}_O = (\mathbf{x} - \mathbf{O}) \times \mathbf{f}$$

- If the origin is $\mathbf{O}$ this reduces to $\mathbf{n} = \mathbf{x} \times \mathbf{f}$.
- If $\mathbf{n}$ is moment about the origin, and n_l is moment about l, and l passes through the origin,

$$n_l = \hat{\mathbf{l}} \cdot \mathbf{n}$$

Total force and moment

- Consider a rigid body, and a system of forces $\{\mathbf{f}_i\}$ acting at $\{\mathbf{x}_i\}$ resp.

Definition

The total force **F** is the sum of all external forces.

$$\mathbf{F} = \sum \mathbf{f}_i$$

Definition

The total moment **N** is the sum of all corresponding moments.

$$\mathbf{N} = \sum \mathbf{x}_i \times \mathbf{f}_i$$

Equivalent systems of forces

- We now develop some equivalence theorems, comparable to (or dual to) our earlier results in kinematics.

Definition

Two systems of forces are equivalent if they have equal total force **F** and total moment **N**.

- Equivalent, specifically, because they would have the same effect on a rigid body, according to Newton.

Resultant

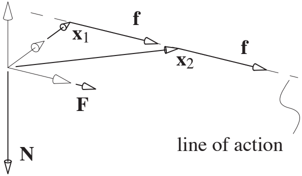
Definition

The resultant of a system of forces is a system comprising a single force, equivalent to the given system.

- A question: does every system of forces have a resultant?

Line of action

- Consider a force $\mathbf{f}$ applied at some point $\mathbf{x}_1$.
- Total force: $\mathbf{F} = \mathbf{f}$
- Total moment: $\mathbf{N} = \mathbf{x}_1 \times \mathbf{f}$.



- Consider line parallel to $\mathbf{f}$ through $\mathbf{x}_1$, and a second point $\mathbf{x}_2$ on the line.
- Force $\mathbf{f}$ through $\mathbf{x}_2$ is *equivalent* to force $\mathbf{f}$ through $\mathbf{x}_1$.
- So *point* of application is more than you need to know . . .

Definition

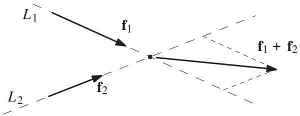
The line of action of a force is a line through the point of application, parallel to the force.

Bound and free vectors

- When you first learned about vectors (in high school?) you learned they aren't attached anywhere. We refer to those as free vectors.
- We can also define bound vectors, specifically a vector bound to a point, called a point vector, and a vector bound to a line, called a line vector.
- So a force is a line vector.

Resultant of two forces

- Let $\mathbf{f}_1$ and $\mathbf{f}_2$ act along L_1 and L_2 respectively.
- Slide $\mathbf{f}_1$ and $\mathbf{f}_2$ along their respective lines of action to the intersection (if any)
- Resultant: the vector sum $\mathbf{f}_1 + \mathbf{f}_2$, acting at the intersection.



- So *almost* every system of forces in the plane has a resultant. Sort of like how almost every motion is a rotation. Can it be extended? Does every system of forces have a resultant?

Change of reference

Using reference Q or R, a system is described by

$$\mathbf{F}_Q = \sum f_i \qquad \mathbf{N}_Q = \sum (\mathbf{x}_i - \mathbf{Q}) \times \mathbf{f}_i$$
$$\mathbf{F}_R = \sum f_i \qquad \mathbf{N}_R = \sum (\mathbf{x}_i - \mathbf{R}) \times \mathbf{f}_i$$

From which it follows

$$\mathbf{F}_R = \mathbf{F}_Q$$
$$\mathbf{N}_R - \mathbf{N}_Q = \sum (\mathbf{Q} - \mathbf{R}) \times \mathbf{f}_i$$

which gives

$$\mathbf{N}_R = \mathbf{N}_Q + (\mathbf{Q} - \mathbf{R}) \times \mathbf{F}$$

Couple

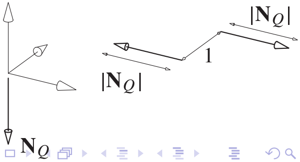
- Is a moment like a force? Can you apply a moment? Does it have a line of action?

Definition

A couple is a system of forces whose total force $\mathbf{F} = \sum \mathbf{f}_i$ is zero.

- So a couple is a pure moment.
- Notice that the moment $\mathbf{N}$ of a couple is independent of reference point. $\mathbf{N}$ is a free vector.
- Does a couple have a resultant? No! This answers the previous question: Not every system of forces has a resultant.

- For an arbitrary couple, can you construct an equivalent system of just two forces?



Equivalence theorems

- Our goal: to define a *wrench*, and show that every system of forces is equivalent to a wrench.
- Analogous to the earlier notes on kinematics, resulting in definition of *twist*.

Theorem

For any reference point Q, any system of forces is equivalent to a single force through Q, plus a couple.

Proof.

- Let $\mathbf{F}$ be the total force;
- let $\mathbf{N}_Q$ be the total moment about Q.
- Let new system be $\mathbf{F}$ at Q, plus a couple with moment $\mathbf{N}_Q$.

□

Two forces are sufficient

Theorem

Every system of forces is equivalent to a system of just two forces.

Proof.

- Given arbitrary **F** and **N**, construct equivalent force and couple, comprising three forces in total.
- Move couple so that one of its forces acts at same point as **F**.
- Replace those two forces with their resultant.

□

Planar system with nonzero **F** has a resultant

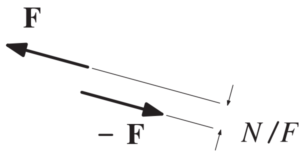
Theorem

A system consisting of a single non-zero force plus a couple in the same plane, i.e. a torque vector perpendicular to the force, has a resultant.

Proof.

- Let **F** be the force, acting at P.
- Let **N** be the moment of the couple.
- Construct an equivalent couple as in the figure.
- Translate the couple so $-\mathbf{F}$ is applied at P.

□



Poinsot's theorem

Theorem (Poinsot)

Every system of forces is equivalent to a single force, plus a couple with moment parallel to the force.

Proof.

- Let $\mathbf{F}$ and $\mathbf{N}$ be the given force and moment. We can assume nonzero $\mathbf{F}$, else the theorem is trivially true.
- Decompose the moment: $\mathbf{N}_{\parallel}$ parallel to $\mathbf{F}$, and $\mathbf{N}_{\perp}$ perpendicular to $\mathbf{F}$.
- Since planar system with nonzero force has a resultant, replace $\mathbf{F}$ and $\mathbf{N}_{\perp}$ by a single force $\mathbf{F}'$ parallel to $\mathbf{F}$.
- The desired system is $\mathbf{F}'$ plus a couple with moment $\mathbf{N}_{\parallel}$.

□

Wrench

Definition

A wrench is a screw plus a scalar magnitude, giving a force along the screw axis plus a moment about the screw axis.

- The force magnitude is the wrench magnitude, and the moment is the twist magnitude times the pitch.
- Thus the pitch is the ratio of moment to force.
- Poinsot's theorem is succinctly stated: every system forces is equivalent to a wrench along some screw.

Screw coordinates for wrenches

- Let f be the magnitude of the force acting along a line l,
- Let n be the magnitude of the moment about l.
- The magnitude of the wrench is f.
- Recall definition in terms of Plücker coordinates:

$$\mathbf{w} = f\mathbf{q}$$
$$\mathbf{w}_0 = f\mathbf{q}_0 + fp\mathbf{q}$$

where $(\mathbf{q}, \mathbf{q}_0)$ are the normalized Plücker coordinates of the wrench axis l, and p is the pitch, which is defined to be

$$p = n/f$$

Screw coordinates for wrenches demystified

- Let $\mathbf{r}$ be some point on the wrench axis

$$\mathbf{q}_0 = \mathbf{r} \times \mathbf{q}$$

- With some substitutions ...

$$\mathbf{w} = \mathbf{f}$$
$$\mathbf{w}_0 = \mathbf{r} \times \mathbf{f} + \mathbf{n}$$

- which can be written:

$$\mathbf{w} = \mathbf{f}$$
$$\mathbf{w}_0 = \mathbf{n}_0$$

  where $\mathbf{n}_0$ is just the moment of force at the origin.
- Screw coordinates of a wrench are actually a familiar representation $(\mathbf{f}, \mathbf{n}_0)$.
- Wrenches form a vector space. You can scale and add them, just as with differential twists.

Reciprocal product of twist and wrench

Reciprocal product:

$$(\omega, \mathbf{v}_0) * (\mathbf{f}, \mathbf{n}_0) = \mathbf{f} \cdot \mathbf{v}_0 + \mathbf{n}_0 \cdot \omega$$

The power produced by the wrench $(\mathbf{f}, \mathbf{n}_0)$ and differential twist $(\omega, \mathbf{v}_0)$.

A differential twist is reciprocal to a wrench if and only if no power would be produced.

Repelling if and only if positive power.

Contrary if and only if negative power.

Force versus motion

- Wrench coordinates and twist coordinates seem to use different conventions:
  - For twists, rotation is first. For wrenches, the opposite.
  - For twists, pitch is translation over rotation, the opposite.
- But these seeming inconsistencies are *not* a peculiar convention. They reflect deep differences between kinematics and statics. For example, consider the meaning of screw axis—the line—in kinematics and in statics. In kinematics, it is a *rotation* axis. In statics, it is a line of *force*.

Comparing motion and force

Motion

A zero-pitch twist is a pure rotation.

For a pure translation, the direction of the axis is determined, but the location is not.

A differential translation is equivalent to a rotation about an axis at infinity.

In the plane, any motion can be described as a rotation about some point, possibly at infinity.

Force

A zero-pitch wrench is a pure force.

For a pure moment, the direction of the axis is determined, but the location is not.

A couple is equivalent to a force along a line at infinity.

In the plane, any system of forces reduces to a single force, possibly at infinity.

Zero Moment Point

The Zero Moment Point (ZMP) is a concept used in legged locomotion. For the simplest cases of a horizontal support surface, and a humanoid standing at rest, the ZMP coincides with the Center of Pressure, where the resultant of contact forces intersects the ground plane, inside the support polygon.