# 15-859N Spectral Graph Theory and The Laplacian Paradigm, Spring 2020

**Homework 1 Version 1.1** **Due: Friday January 31 in class**

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**Instructions.** Collaboration is permitted in groups of size at most three. You must write the names of your collaborators on your solutions and must write your solutions alone.

| Question | Points | Score |
|---|---|---|
| 1 | 25 | |
| 2 | 25 | |
| Total: | 50 | |

(25) 1. **Pseudoinverses**

Wikipedia defines the Moore-Penrose inverse or Pseudoinverse to be the operation that takes an $n \times m$ matrix $A$ and returns an $m \times n$ $A^+$ satisfying the following constraints. We will assume that $A$ is real this will simpify the constraints to:

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(AA^+)^T = AA^+$
4. $(A^+A)^T = A^+A$

Recall that in class we defined the pseudoinverse of a symmetric real matrix $M$ to be $U\Lambda^+U^T$ where $M = U\Lambda U^T$ is the spectral decomposition of $M$.

1. Show that our definition agrees with the Moore-Penrose inverse and it is the only solution.
2. In the case where $A$ is real symmetric do we need all four axioms or constraints?

(25) 2. **Effective Resistance and Selfloops**

## 1 ER from the inverse of Laplacian

Given a connected graph of conductors $G = (V, E, c)$ (for simplicity assume $V = [n]$), let $L$ be its Laplacian. Fix a destination $n \in V$, and let $L_n$ be obatined from $L$ by deleting the last row and last column.

1. Prove that $L_n$ has full rank.
2. Prove that $(L_n^{-1})_{i,i}$ is equal to the effective resistance between $i$ and $n$.

# 2 How do self-loops affect hitting times?

Given a connected graph $G = (V, E, c)$ (for simplicity assume $V = [n]$), let $L$ be its Laplacian. Fix a destination $n \in V$. We defined the hitting time $h_i$ to be the expected number of steps we need to take from $i$ to $n$. For any $i \neq n$, we have $h_i = 1 + \sum_j \frac{c_{i,j}}{c_i} h_j$ with $h_n = 0$, which can be modeled written as $L_n h = [c_1, ..., c_n]^T$ where $L_n$ is obtained from $L$ by deleting the last row and column.

1. Suppose that we add a self-loop of positive weight at some vertices of $G$ to make $G'$. Show that the hitting time is still a solution of $L_n h = [c'_1, ..., c'_{n-1}]^T$ where $c'_i \geq c_i$ for every $i$. Note that $L_n$ is still obtained from the Laplacian of $G$.
2. Intuition tells that adding a self-loop should increase the hitting time of every vertex. Prove it formally by showing that if $h, h'$ satisfy $L_n h = c$ and $L_n h' = c'$ where $c \leq c'$ (which means that $c_i \leq c'_i$ for all $i$), $h \leq h'$.
3. Does the above fact also holds when $L_n$ is replaced by any other positive-definite matrix? In other words, if $A$ is a symmetric, positive definite matrix (so it is invertible), $b \leq b'$, and $Ax = b, Ax' = b'$, is it true that $x \leq x'$?