15-859N — Spectral Graph Theory and Numerical Linear Algebra. — Spring 2020

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Assignment 0 Due date: January 20, 2020

This homework is just a list of fundamantal linear algebra facts you should know. You should have an idea how the proofs also go.

1 Orthogonal Projection

Suppose that $Ax = b$ is an over constrained linear system and we would like to find an x to minimize $|Ax - b|_2^2$, the L_2^2 distance, where the columns of A are independent.

1. Show that the answer to our minimization problem is:
   1) The system $A^T A\bar{x} = A^T b$ always has a solution.
   2) The $\bar{x}$ is solution to our problem.
2. The projection of $b \in \mathbb{R}^m$ onto the column space of an m by n matrix A is the linear matrix $A(A^T A)^{-1} A^T b$.

2 Spectral Theorem

Suppose that A is a symmetric n by n real matrix. Show that A has the following properties:

1. The eigenvalues are all real.
2. A has a complete set of eigenvalues and eigenvectors, i.e., Its eigenvectors span a space of dimension n.
3. $A = U^T \Lambda U$ where the rows of U are an orthonormal set of eigenvectors for A and Λ is a diagonal matrix of eigenvalues for A.
4. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and $x_1, \ldots, x_n$ are the respective orthonormal eigenvectors as column vectors then

$$A = \lambda_1 x_1 x_1^T + \cdots + \lambda_n x_n x_n^T$$

3 Matrix Exponential

Assuming that A is real symmetric, use the Spectral Theorem show that e^A is well defined and give a simple expression for it. That is, how do the eigenvalues and eigenvectors of A relate to those of e^A?