Ground Rules

The homework consists of a few exercises followed by some questions. The exercises will not be graded, and are given to help you better understand the course material. Please let us know, for each question, if you have seen the question before. And do prove claims that you make.

This assignment (and the next one) should be done individually: **no collaboration is allowed**. Also, we will be strict with the deadline on these two: **this one is due in class on Monday Nov 15th.**

Questions

1. **(Rewrites.)**

   I have n cards, each with a distinct number written on it. I shuffle them perfectly, and go over these cards in the resulting uniform random order: every time I see a number bigger than all the ones I have seen before, I write it down. (Yes, I write down the very first number I see.)

   How many numbers do I expect to write down during this process?

2. **(Azuma's Inequality.)**

   Let $\mathfrak{S}_n$ denote the set of all permutations on n elements $\{1, 2, \ldots, n\}$. For $\pi \in \mathfrak{S}_n$, let

   $$I(\pi) = |\{(i, j) \mid 1 \le i < j \le n \text{ and } \pi_i > \pi_j\}| \tag{1}$$

   be the number of *inversions* in π.

   (a) Let π be drawn uniformly at random from $\mathfrak{S}_n$, and let X denote the number of inversions in π. What is EX?

   (b) For any $i \in \{1, 2, \ldots, n\}$, let Π_i be a partition of $\mathfrak{S}_n$ where each class of Π_i consists of permutations that agree in positions 1 through i. I.e., π and π' are in the same class of Π_i iff $\pi_j = \pi'_j$ for all $j \le i$. Show that each set in Π_i has size $(n - i)!$.

   (c) Let $\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots \subseteq \Sigma_n$ be the filter generated by the partitions Π_i. If $Z_i = E[X \mid \Sigma_i]$, then show that $|Z_i - Z_{i-1}| \le 2n$. Infer that the number of inversions in a random permutation is $EX(1 + o(1))$ **whp**.

3. **(Lovasz Local Lemma and Colorings.)**

   Consider an undirected graph $G = (V, E)$, where each vertex v has a set $S(v)$ of colors. A list-coloring χ of G assigns each vertex $v \in V$ a color from its set $S(v)$. A proper list-coloring is a list-coloring that ensures that all edges are bichromatic.

   Suppose each vertex has a list of size $10k$. Moreover, for each $v \in V$ and $c \in S(v)$, there are at most k neighbors u of v that contain c in their color sets $S(u)$. Show that there exists a proper list-coloring of G with these parameters.

4. **(Random Walks.)**

   Show that the cover time of a random walk on a connected d-regular graph is $O(n^2 \log n)$. (Note that this quantity is independent of d.)

5. **(A Paul and Carole Game.)**

Consider the following perfect information game between two players, Paul (partitioner) and Carole (chooser). Let $\vec{P} \in \Re^n$ be a vector, originally set to 0^n. There are n rounds: in round i, Paul first selects a vector $\vec{v_i} \in \{-1, 1\}^n$. Carole then resets $\vec{P}$ to either $\vec{P} + \vec{v_i}$ or $\vec{P} - \vec{v_i}$. Let $\vec{P}^*$ denote the final value of $\vec{P}$ after n rounds. The payoff to Paul is $\|\vec{P}^*\|_\infty$, the largest over absolute values of the coordinates of $\vec{P}^*$. We say that Paul wins if this value is at least α, else Carole wins. This is called the (α, n) *balancing vector game.*

(a) Let S_n denote the random variable that is the sum of n independent random variables X_i, with each $X_i \in_R \{-1, 1\}$. Prove that if $n \Pr[|S_n| \geq \alpha] < 1$, then Carole has a winning strategy for the (α, n) balancing vector game.

(b) Derandomize the above argument to give a deterministic strategy for Carole.