15-859(M): Randomized Algorithms **Homework 1**
Date: Sept 20, 2004 **Due:** Oct 4, 2004

Ground Rules

The homework consists of a few exercises followed by some questions. The exercises will not be graded, however, we encourage you to solve them for a better understanding of the course. You are allowed to work in small groups, but must turn in solutions individually. Please let us know, for each question, if you worked in a group, or if you have seen the question before.

Exercises

1. **A Classic.** You are given a biased coin with a fixed but unknown bias p. How would you simulate an unbiased coin (i.e., one with bias $\frac{1}{2}$) using this coin?

2. **Balanced Partitions.** You want to sample uniformly from the following set:

$$A = \{x \in \{0,1\}^{2n} \mid x \text{ has an equal number of 0's and 1's } \} \tag{1}$$

   Given an unbiased coin, how would you generate a random element of this set? (You should strive to use as little randomness as possible.)

3. **Complexity Classes.** In class we showed that ZPP $\subseteq$ RP $\cap$ co-RP. Prove that RP $\cap$ co-RP $\subseteq$ ZPP, implying ZPP $=$ RP $\cap$ co-RP.

4. **Evaluating Majority gates.** Consider a uniform rooted tree of height h, where each internal node, including the root, has 3 children. Each node returns the value returned by the majority over its children. The problem is to evaluate the value at the root, given values at each of the leaves. Let $n = 3^h$ be the number of leaves in the tree.

   (a) Show that any deterministic algorithm for this problem must read all the n leaf nodes in the worst case.

   (b) Show that a non-deterministic algorithm can solve this problem in $n^{\log_3 2}$ time. That is, for any input, there is a set of $n^{\log_3 2}$ leaves whose value determines the value at the root.

   (c) Consider the randomized algorithm that, for every internal node, evaluates two of its children, and evaluates the third only if the first two answers are different. Show that the expected number of leaves read by this algorithm is $n^{0.9}$.

Questions

1. **More coin flipping.**

   (a) Given an unbiased coin, how would you simulate a coin with bias p? (I.e., you must say "Heads" with probability p, and "Tails" with probability $1 - p$.) Try to minimize the expected number of coin tosses.

   (b) Given an unbiased coin and n elements, how can one generate a random permutation of these elements?

   (c) You are given a stream of elements that are whizzing past you, and you have storage enough to hold only k elements. The goal is to maintain a random sample of k elements from the stream.

I.e., when you have seen t of the stream elements, the storage must contain a random one of the $\binom{t}{k}$ possible k-subsets; moreover, this should hold for all times $t \geq k$.

How would you do this? (Note that if you do not put an element into storage, it is gone forever.)

2. **More on Max-Cut.** Recall that m is the number of edges in a graph.

   (a) **Randomized Max-Cut.** Improving on the bound proved in Lecture 1, prove that every graph has a cut containing at least $\frac{m}{2}(1 + \frac{1}{n})$ edges.

   (b) **Planar Max-Cut.** Prove that any planar graph contains a cut containing at least $\frac{2m}{3}$ edges. (Hint: Use the fact that planar graphs are 4-colorable.)

3. **Perfect matchings in general graphs.** In this question, you will use ideas discussed in Lecture 3 to give an algorithm for determining whether an arbitrary graph G contains a perfect matching or not.

   The Tutte matrix M of a graph $G = (V, E)$ is given as follows.

$$M_{ij} = \begin{cases} x_{ij} & \text{if } (i,j) \in E \text{ and } i < j \\ -x_{ij} & \text{if } (i,j) \in E \text{ and } i > j \\ 0 & \text{otherwise} \end{cases}$$

   (a) Write down an expression for the determinant of M.

   (b) Prove that if there is a perfect matching in G, then $\mathrm{Det}(M)$ is non-zero. (Hint: Give explicit values of x_{ij}s.)

   (c) The sign of a permutation is the parity of the number of transpositions required to transform it to the identity permutation. (E.g.: The sign of $(1, 2, 3)$ is 1, whereas that of $(3, 2, 1)$ is -1.) Show that the sign of a permutation is the same as the sign of its inverse.

   (d) Use part (c) of the question to argue that the non-zero terms in the determinant correspond to perfect matchings in G. Conclude that $\mathrm{Det}(M) \neq 0$ iff there is a perfect matching in G.

4. **A number game.** (Due to T. Cover and M. Rabin). Consider the following game. A friend writes down two arbitrary numbers on two slips of paper and then randomly puts one in one hand and the other in the other hand. You get to pick a hand and see the number in it. You then can either keep the number you saw or else return it and get the other number. Say you end up with the number a and the other number was b. Then, your gain is $a - b$.

   For a given (possibly randomized) strategy S, let $\mathbf{E}[S \mid x, y]$ denote its expected gain, given that the two numbers that your friend has in mind are x and y. (Note that the expectation is over the randomness in the strategy, and over the randomness in picking a hand — you see x with probability $\frac{1}{2}$ and y with probability $\frac{1}{2}$.)

   (a) Consider the strategy $S =$ "if the first number I see is ≥ -17, then I keep it, else I switch." What is $\mathbf{E}[S \mid x, y]$ in terms of x and y?

   (b) Give a randomized strategy S such that $\mathbf{E}[S \mid x, y] > 0$ for all $x \neq y$.