Sparsest Cut Generalizations

15-750
4/11/16

So far we have considered 2 cut probs.

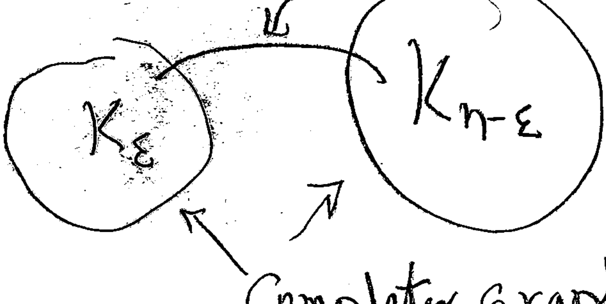
1) S-t-Min Cut
2) Separators (Edge version)

Both wanted small # edges give a vertex partition A, B.

1) required $s \in A$ & $t \in B$
2) $|A|$ & $|B|$ not too big. (Strongly Balanced)

Goal: Consider relaxations of partition condition!

Single edge e



Complete graphs

1) $G=(V,E)$ network

2) All nodes communicate to all other nodes.

Find network bottle neck!

i.e. $$\min_{S \subseteq V} \frac{|\mathrm{cut}(S,\bar{S})|}{|S|\cdot|\bar{S}|}$$ (Sparsest Cut) (Uniform)

Here the communication pattern K_n (the complete graph)

$$\sigma_G(S) = \frac{|\mathrm{Cut}_G(S,\bar{S})|}{|\mathrm{Cut}_H(S,\bar{S})|} \qquad H = K_n$$

$\min\limits_{S} \sigma_G(S)$ more generally $\sigma(G,H) = \min\limits_{S} \sigma_{G,H}(S)$

note G flow graph & $H \equiv \underset{s}{\circ}\!\!-\!\!\underset{t}{\circ}$ then

$\sigma(G,H) \equiv$ min s-t cut.

As in Min s-t Cut lets consider assigning length to edges of G, say d_{uv}.

Given d over edges to all pairs by shortest path.

Recall: Metric $d: V \times V \to \mathbb{R}$

1) $d(x,y) \geq 0$
2) $d(x,y) = d(y,x)$
3) $d(x,z) \leq d(x,y) + d(y,z)$ Δ ineq
4) $d(x,y) = 0$ iff $x = y$

Def d is semi-metric if 1), 2), 3)

Leighton-Rao Relaxation

$$LR(G,H) = \min_{d \text{ semi-metric}} \frac{\sum_{uv \in G} d_{uv}}{\sum_{uv \in H} d_{uv}}$$

Simplest semi-metric on V

$f: V \to \{0,1\}$

$d_{uv} = |f(u) - f(v)|$

$$\underline{\text{Sparsest Cut}} \equiv \min_{f: V \to \{0,1\}} \frac{\sum_{uv \in G} |f(u)-f(v)|}{\sum_{uv \in H} |f(u)-f(v)|} = \nabla(G,H)$$

$\underline{\text{Thus}}$ $LR(G,H) \le \nabla(G,H)$

$\underline{\text{Thm}}$ $\nabla(G,H) \le O(\log n) \cdot LR(G,H)$

What LP for Leighton-Rao

$\underline{\text{Claim}}$ $\quad \min \sum_{uv \in G} d_{uv}$

Subject: $\sum_{uv \in H} d_{uv} = \frac{|E_G|}{|E_H|}$

$d_{uv} \le d_{uw} + d_{wv} \quad \forall u,v,w \in V$

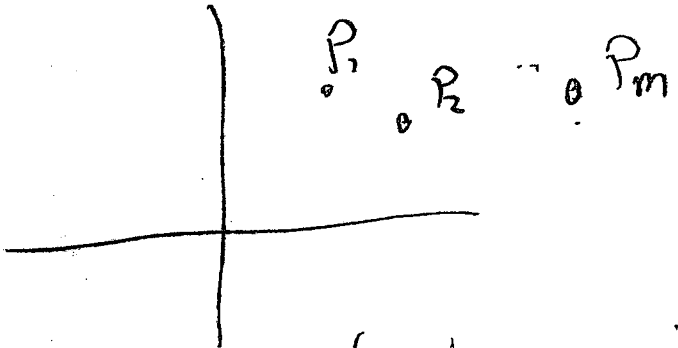
$d_{uv} \ge 0 \quad \forall u,v \in V$

Simple fact

Lemma $a_1, \dots, a_m \geq 0$ & $b_1, \dots, b_m > 0$ then

$$\max_i \frac{a_i}{b_i} \geq \frac{a_1 + \dots + a_m}{b_1 + \dots + b_m} \geq \min_i \frac{a_i}{b_i}$$

pf consider $\left(\frac{a_i}{b_i}\right) \equiv$ point in $\mathbb{R}^2$ $(b_i, a_i) = P_i$



$$\text{mean}(P_1 \cdots P_m) = \left(\sum b_i, \sum a_i\right)/m$$

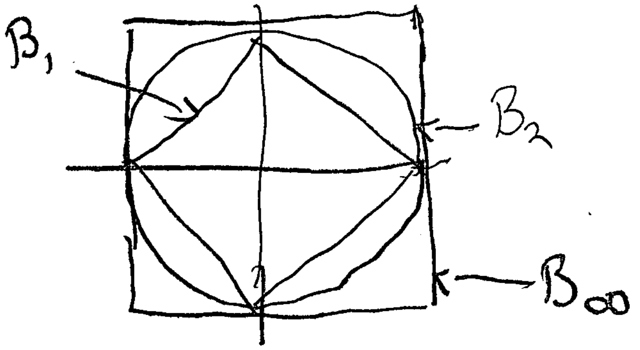
$$\text{slope}(\text{Line}(0, P_i)) = \left(\frac{a_i}{b_i}\right)$$

Slope of mean $\geq$ min slope

L_1 Relaxation of Sparsest Cut

$x, y \in \mathbb{R}^n \quad \|x\|_1 = \sum_i |x_i| \quad$ metrix $\|x-y\|_1 = \sum_i |x_i - y_i|$

L_1, L_2, L_∞ in $\mathbb{R}^2 \quad B_i = \{x \in \mathbb{R}^2 \mid \|x\|_{L_i} = 1\}$



$$\nabla_1(G,H) = \inf_{m, f: V \to \mathbb{R}^m} \frac{\sum_{uv \in G} \|f(u) - f(v)\|_1}{\sum_{uv \in H} \|f(u) - f(v)\|_1}$$

<u>Thm</u> $\forall G, H \quad \nabla(G,H) = \nabla_1(G,H)$

<u>pf</u> Clearly $\nabla_1(G,H) \le \nabla(G,H)$

to Show $\quad \nabla(G,H) \le \nabla_1(G,H) \qquad (*)$

proof of (*)

$$f(v) = (f_1(v), \cdots, f_m(v))$$

$$a_i = \sum_{uv \in G} |f_i(u) - f_i(v)| \quad b_i = \sum_{uv \in H} |f_i(u) - f_i(v)|$$

$$\pi_1(G,H) = \frac{\sum a_i}{\sum b_i} \geq \min_i \frac{a_i}{b_i}$$

Thus WLOG $m = 1$ ie $f: V \to \mathbb{R}$

π is invariant under translation & scaling

$$\min_v f(v) = 0 \quad \& \quad \max_v f(v) = 1$$