More NP-Completeness

15-750
4/23/10

Graph Coloring

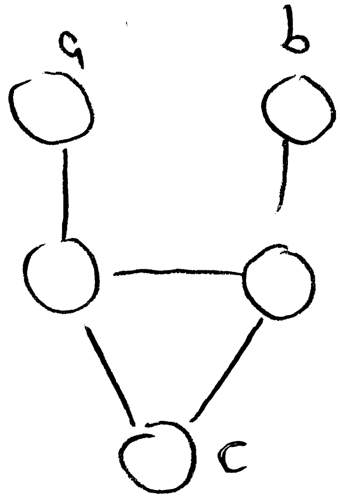
Through out $G=(V,E)$ undirected

Def A K-coloring is a fcn $C: V \to \{0,1,\dots,K-1\}$ s.t. the end-points of each edge have distinct colors.

Def 3-Color $\equiv \{G \mid G$ is 3-colorable$\}$

Thm 3-Color is NP-Complete

the color gadget:



$C \equiv$ 3-coloring

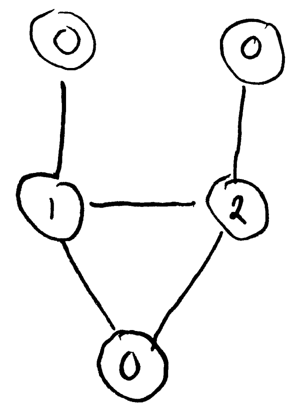
Colors $=\{0,1,2\}$

Critical Facts

1) $\forall C$ if $C(a) = C(b)$ then $C(a) = C(c)$

pf WLOG $C(a) = C(b) = 0$

Thus

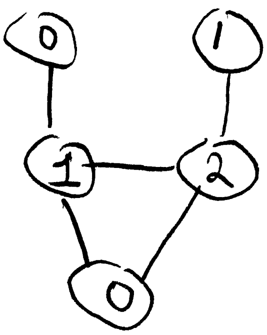


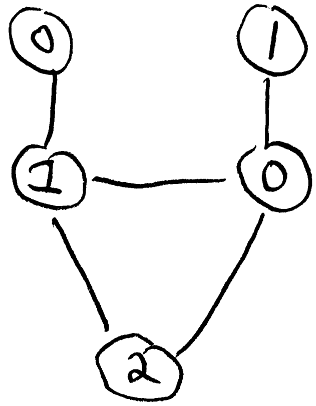
2) $\forall x \neq y, z \in \{0,1,2\}\ \exists C$ st

$C(a) = x \wedge C(b) = y \wedge C(c) = z$

WLOG $x = 0 \wedge y = 1$

WLOG $z \in \{0, 2\}$





Claim $3CNF \leq_p 3\text{-Color}$

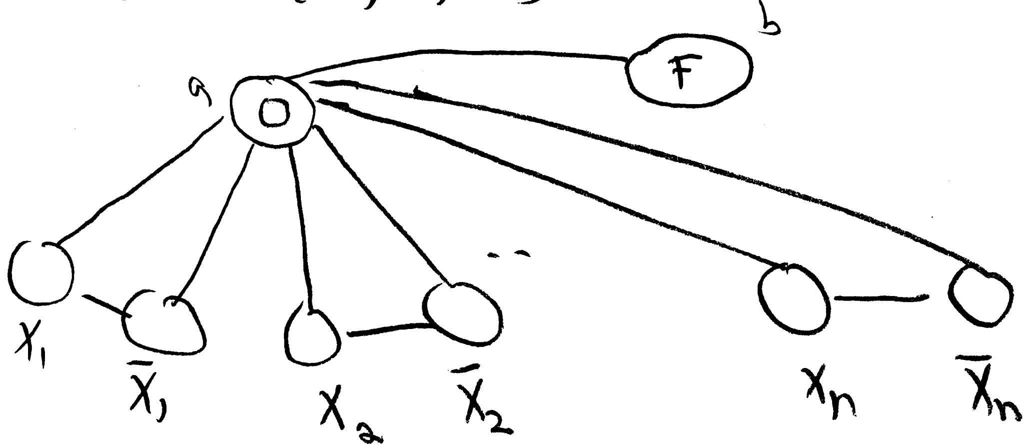
The construction of $f(\varphi)$

φ has clauses $C_1 \cdots C_K$

Variables $x_1 \cdots x_n$

A color wheel for each variable

Colors $= \{0, T, F\}$

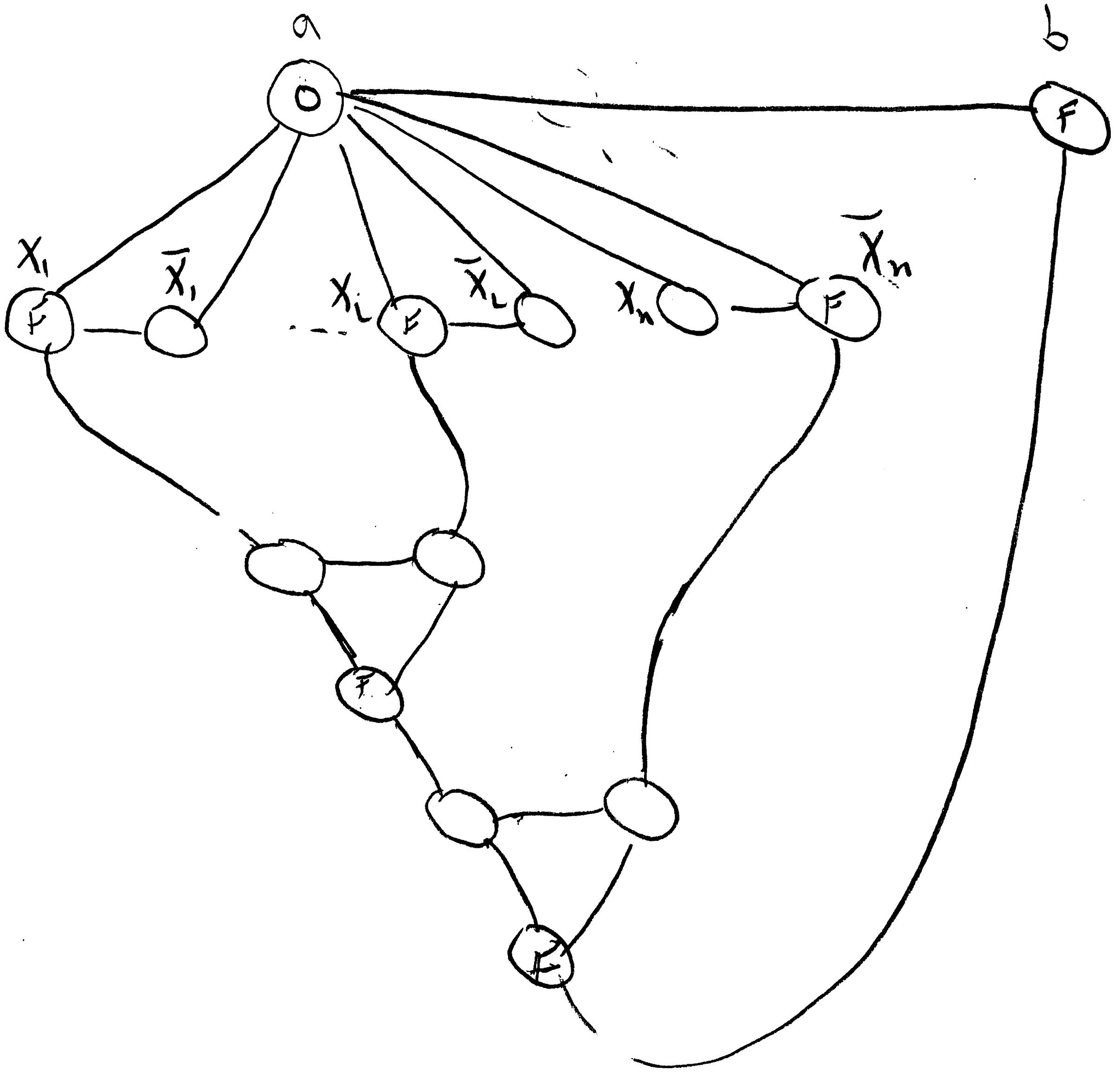


Our Color Wheels

WLOG $C(a) = 0$ & $C(b) = F$

Attaching 3-Clauses to Color Wheels

eg $C_j = (x_1 \vee x_i \vee \bar{x}_n)$



Add one for each clause

<u>Claim</u> φ is sat iff $G = f(\varphi)$ is 3-colorable

($\Rightarrow$) 1) Use truth assignment to color color-wheels with colors $\{0, T, F\}$

2) color gadgets

$\Leftarrow$ View $C(a) = 0$ & $C(B) = F$ & remaining color as T.

Use colors of $x_1 \cdots x_n$ as truth assignment.

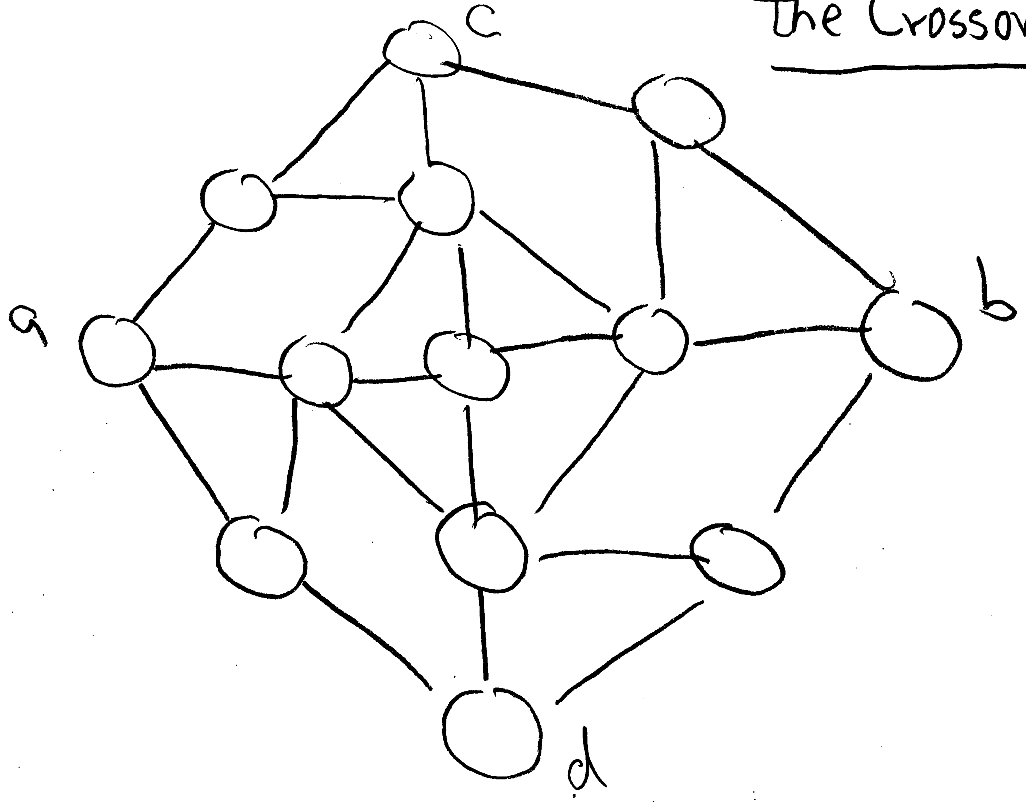
<u>Def</u> Planar-k-color $= \{ G \text{ planar} \mid G\ k\text{-colorable} \}$

<u>Thm</u> Planar-4-color $\in P$

<u>Thm</u> Planar-3-color is NP-Complete

The gadget of all gadgets!

the Crossover gadget



1) If C is a 3-coloring then
$$C(a) = C(b) \ \& \ C(c) = C(d)$$

2) Given $X, Y \in \{0,1,2\}$ $\exists C$ st
$$X = C(a) = C(b) \ \& \ Y = C(c) = C(d)$$

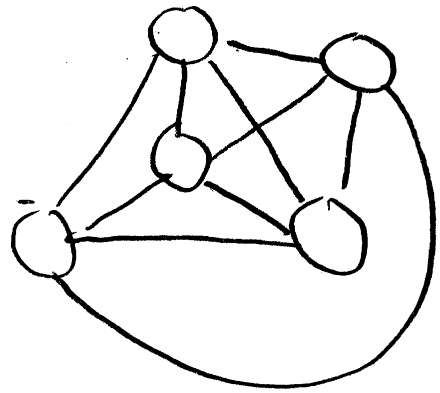
Given $G=(V,E)$ Goal construct $f(G)$

1) "Draw" G in plane with crossings

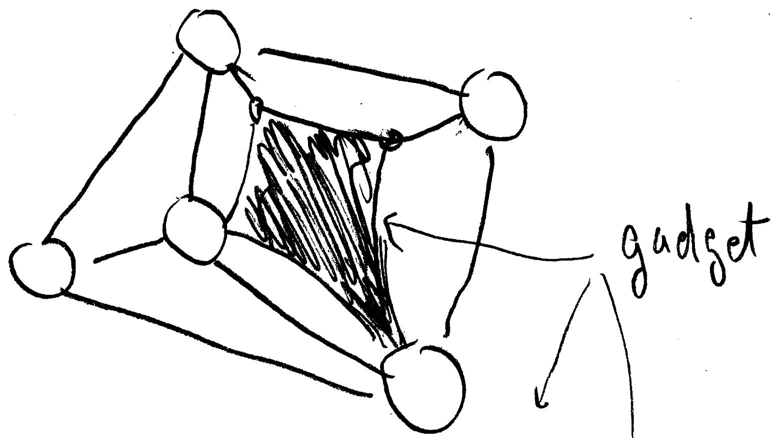
2) Insert a crossover gadget for each crossing

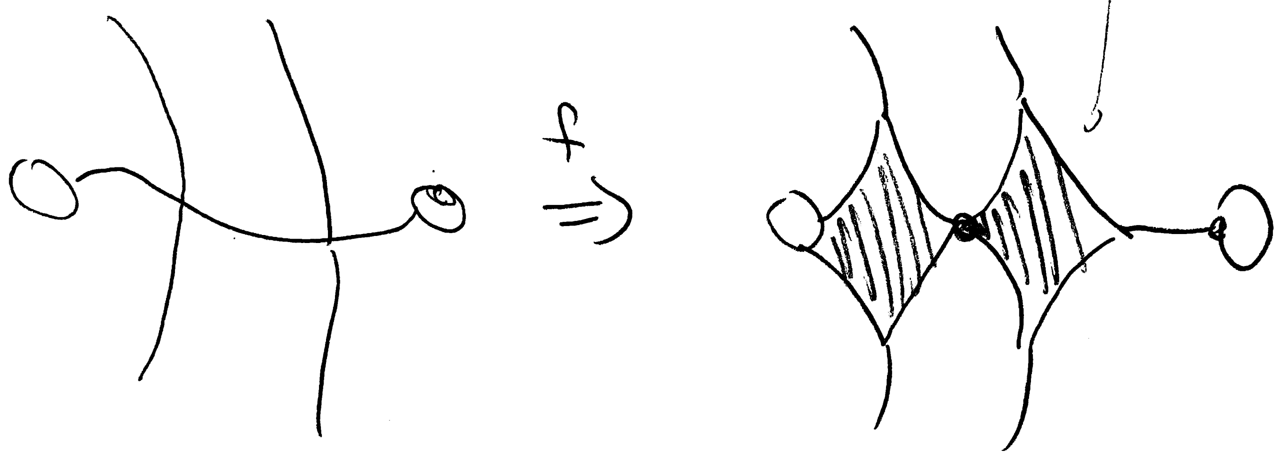
eg K

$G \equiv$



$f(G) \equiv$





Exact Cover

Input finite set X, Subsets $\{S_1, \dots, S_k\} = \mathcal{S}$

Question $\exists \mathcal{S}' \subseteq \mathcal{S}$ s.t. $\mathcal{S}'$ is an exact cover

ie $\forall x \in X \ \exists! \, s \in \mathcal{S}'$ s.t. $x \in s$

Thm Exact Cover is NP-Complete

Claim 3-Color $\leq_p$ Exact Cover

$G = (V, E)$ is 3-color prob & $C = \{\text{red}, \text{blue}, \text{green}\}$

Def $N(u) = \{ v \in V \mid (u,v) \in E \}$ $u \in V$

The Finite set X

For each $u \in V$ add four points

$$u, \; p_u^{red}, \; p_u^{blue}, \; p_u^{green}$$

For each $(u,v) \in E$, add six points

$$q_{uv}^{red} \; q_{uv}^{blue}, \; q_{uv}^{green}, \; q_{vu}^{red}, \; q_{vu}^{blue}, \; q_{vu}^{green}$$

The subsets $\mathcal{S}$

$\forall u \in V$ subsets $S_u^{red}, \; S_u^{blue}, \; S_u^{green}$

eg $S_u^{red} =$

$$p_u^{red}$$

$$q_{uv_1}^{red} \; \cdots \; q_{uv_d}^{red}$$

$N(u) = \{v_1, \dots, v_d\}$

Include $\{u, p_u^{red}\} \{u, p_u^{blue}\} \{u, p_u^{green}\}$

$$\{\{q_{uv}^{c}, q_{vu}^{c'}\} \mid (u,v) \in E \wedge c \neq c'\}$$

<u>Claim</u> G is 3-colorable iff $f(G)$ has an exact cover

($\Rightarrow$) Suppose $C: V \to \{red, blue, green\}$

proof by example! suppose

$(u,v) \in E$ $\quad C(u) = red$ & $C(v) = blue$ $\quad d(u) = 5$ & $d(v) = 3$

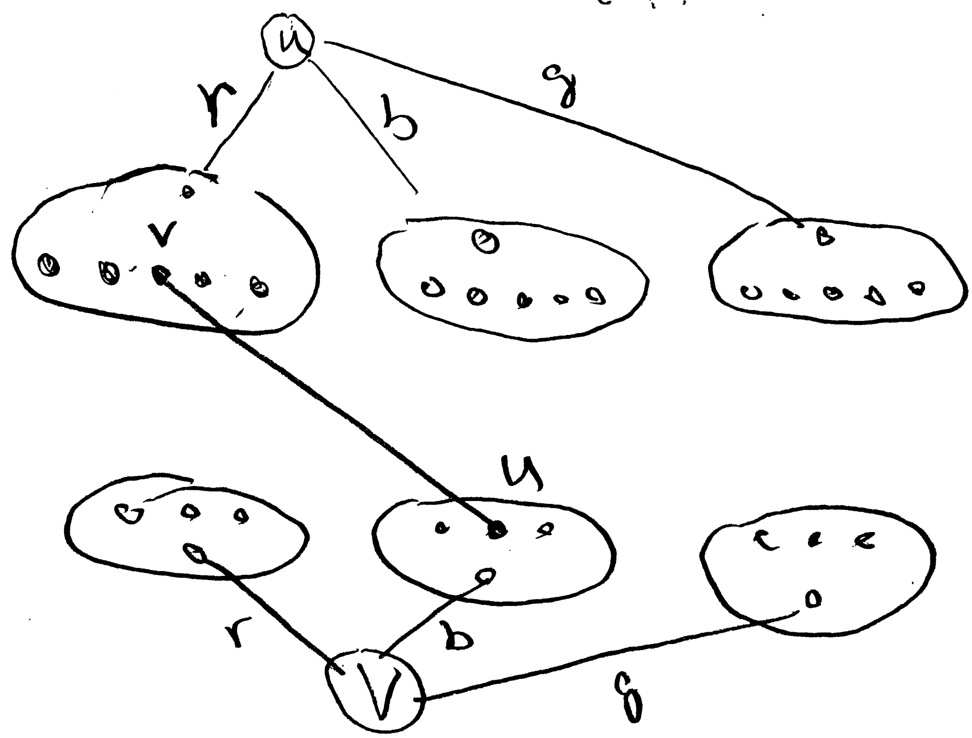
Include in cover

$\{u, p_u^{red}\}\ \{v, p_v^{blue}\}$

$S_u^{c'}$ $\quad c' \neq red$

$S_v^{c'}$ $\quad c' \neq blue$

$\{g_{uv}^{red}, g_{vu}^{blue}\}$



($\Leftarrow$) The subsets $\{u, P_u^c\}$ give a coloring

More NP-Problems

Knapsack

Input Finite set S; cost fcn $w : S \to \mathbb{N}$
benefit fcn $b : S \to \mathbb{N}$
int W, B

Question $\exists S' \subseteq S$ st

$$\sum_{a \in S'} w(a) \leq W \quad \& \quad \sum_{a \in S'} b(a) \geq B$$

Subset Sum

Input Finite set S; $w : S \to \mathbb{N}$; int B

Question $\exists S' \subseteq S$ st

$$\sum_{a \in S'} w(a) = B$$

Partition Prob:

Input: Finite set S; $w: S \to \mathbb{N}$

Question: $\exists S' \subseteq S$ st

$$\sum_{a \in S'} w(a) = \sum_{a \notin S'} w(a)$$

Partition $\leq_p$ Sub set Sum

Set $B = \frac{1}{2} \sum_{a \in S} w(a)$

Partition $\leq_p$ Knapsack

Set $b = w$ & $W = B = \frac{1}{2} \sum_{a \in S} w(a)$

Subset Sum $\leq_p$ Partition

Set S ; $w: S \to \mathbb{N}$; B

Let $W = \sum_{a \in S} w(a) \wedge N >> W$

New elements added to S a & b giving $\bar{S}$

$w(a) = N - B$ & $w(b) = N - W + B$

<u>Claim</u> S has subset sum iff $\bar{S}$ has partition.

($\Rightarrow$) Suppose S' is a subset sum

then $S' \cup \{a\}$ is a partition.

$\Leftarrow$ Suppose $\bar{S}'$ is a partition

if $a \in \bar{S}'$ then $b \notin \bar{S}'$

thus $\bar{S}' - \{a\}$ is a subset sum.

Exact Cover $\leq_p$ Subset Sum

pf Let $(X, \mathcal{S})$ be an exact-cover prob

We construct subset sum prob $(\bar{X}, w)$.

Set $\bar{X} = \mathcal{S}$ (Every subset is an element)

WLOG $X = \{0, 1, \cdots, m-1\}$

for $x \in X$ $\#x = |\{A \in \mathcal{S} \mid x \in A\}|$

pick prime $p > \#x \ \forall x \in X$

Def $w(A) = \sum_{x \in A} p^x$ $B = \sum_{x=0}^{m-1} p^x = \frac{p^m - 1}{p - 1}$

Claim Exact Cover iff Subset Sum

<u>By example</u>

Suppose $A_1 = \{1, 3\}$ $A_2 = \{0, 3\}$

In p-ary notation

$$\begin{array}{lcccccccl} & p^0 & p^1 & p^2 & p^3 & p^4 & \cdots & & p^{m-1} \\ w(A_1) = (& 0, & 1, & 0, & 1, & 0, & \cdots & - & 0 \;) \\ w(A_2) = (& 1, & 0, & 0, & 1, & 0, & \cdots & - & 0 \;) \\ w(A_1)+w(A_2) = (& 1, & 1, & 0, & 2, & 0, & \cdots & - & 0 \;) \end{array}$$

Since $\#3 < p$ the 2 can not generate a carry.