Linear Programming

15-750
2/26/16

Goals:

1) Introduce important "programming Lang"
2) Some examples
3) Introduce its dual program.

Standard forms

<u>Standard form</u> max $c^T x$ subject

$$Ax \le b$$
$$x \ge 0 \quad \text{(new)}$$

$\bar{x}$ s.t. $A\bar{x} \le b$ & $\bar{x} \ge 0$ is <u>feasible solution</u>

a feasible $\bar{x}$ maximizing $c^T\bar{x}$ is called an <u>optimal solution</u>

constraints maybe feasible
infeasible
unbounded

<u>Putting into ~~Standard~~ standard form</u> (possible non-standard)

1) Min $c^T x$ $\quad c \leftarrow -c$

2) missing $x_i \ge 0$ constraint.

3) $a_1 x_1 + \cdots + a_n x_n = b$ (equality constraints)

4) $a_1 x_1 + \cdots + a_n x \ge b$

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2) for each missing $x_i \ge 0$ introduce two variables x_i' x_i''
$x_i', x_i'' \ge 0$

$$a_1 x_1 + \cdots + a_i (x_i' - x_i'') + \cdots + a_n x_n \le b$$

Let $\bar{X}$ be new X varibles.

To show a) Given opt $\bar{X}$ construct opt X

b) " " X " " " $\bar{X}$.

b) if $x_i \geq 0$ then $x_i' = x_i$ & $x_i'' = 0$

else $x_i' = 0$ & $x_i'' = x_i$

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a) Set $x_i = x_i' - x_i''$

New Obj: replace $c_i x_i$ with $c_i x_i' - c_i x_i''$

Note If $\bar{x} = (x_1, \ldots, x_i', x_i'', \ldots)$ is an opt

then $\forall a \geq 0$ $\bar{x}' = (x_1, \ldots, a + x_i', a + x_i'', \ldots)$ is an opt.

LP for Single Source Shortest Path.

Input: weighted directed graph

$G = (V, E)$ $w : E \to \mathbb{R}$, $s \in V$

Output: 1) "Neg weight cycle" or

2) $d : V \to \mathbb{R}$

st $d_v \equiv$ min dist from s to v.

Constraints $d_s = 0$

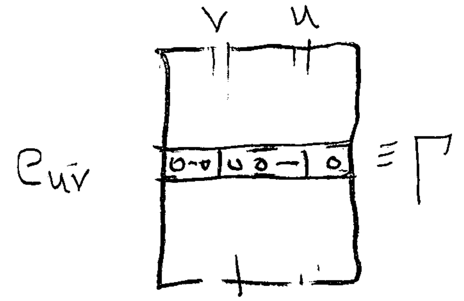
$(u,v) \in E$ then $d_v \le d_u + w_{uv}$

$$\max \sum_{v \in V} d_v$$

$d_v - d_u \le w_{uv}$

$+1 \longrightarrow -1$

Standard form



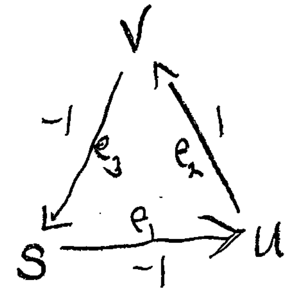
$d_s = d_1 = 0$

$$\Gamma \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \le \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$$

$\mathbb{1} \equiv \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ $\max \mathbb{1}^T d$

Drop first column (why?)

<u>eg</u> negative cycle example

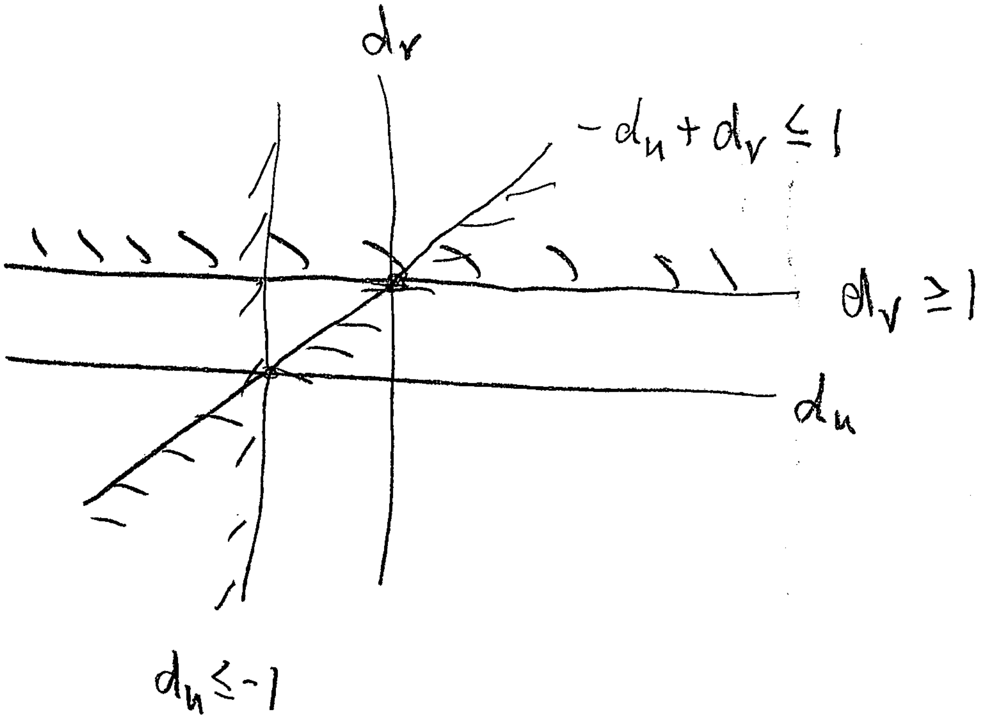
$G =$ 

$$\Gamma = \begin{array}{c} \begin{array}{ccc} s & u & v \end{array} \\ \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \end{array}$$

$$\Gamma \begin{pmatrix} d_s \\ d_u \\ d_v \end{pmatrix} \leq \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

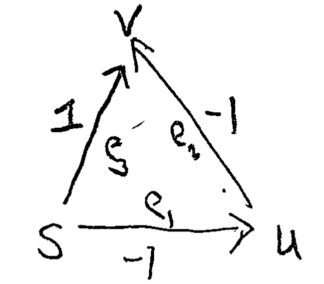
Drop col 1

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} d_u \\ d_v \end{pmatrix} \leq \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$



<u>infeasible</u>

eg $G =$



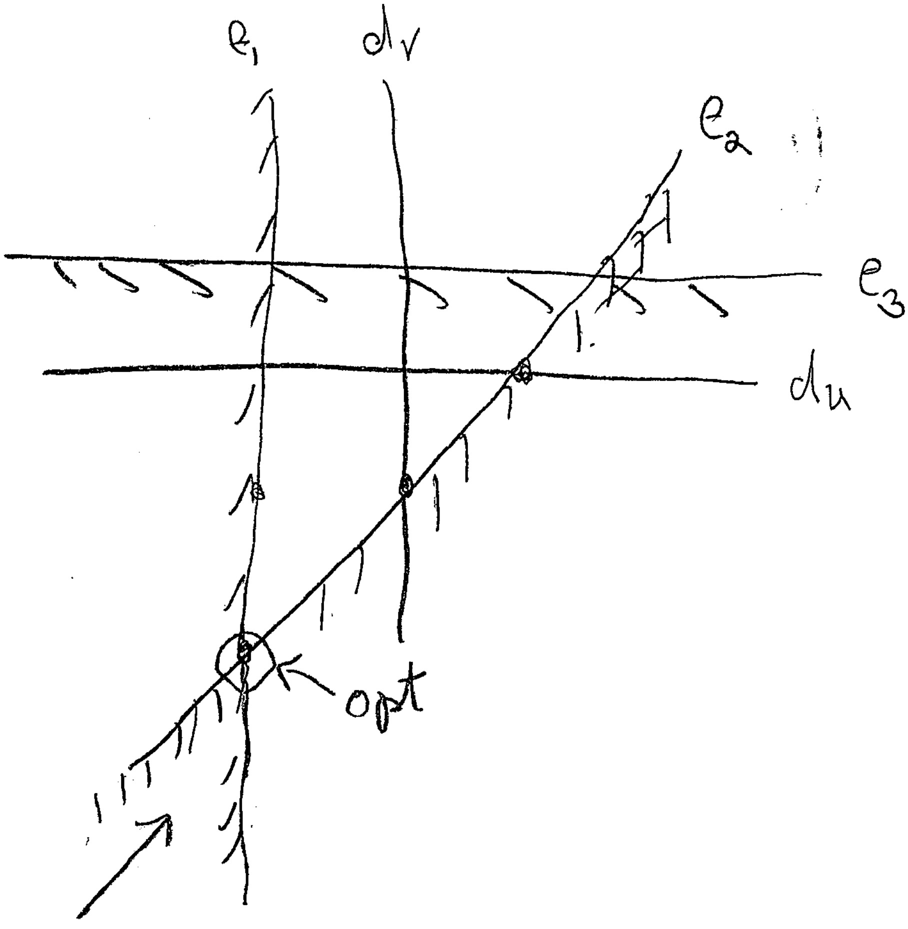
$$\Gamma = \begin{array}{c} \begin{array}{ccc} s & u & v \end{array} \\ \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \end{array}$$

$$\Gamma \begin{pmatrix} d_s \\ d_u \\ d_v \end{pmatrix} \leq \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Drop col 1

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_u \\ d_v \end{pmatrix} \leq \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

max $d_u + d_v$



<u>Note</u> We have not shown that our SSSP-LP gives SSSP!

We will need Duality for that!

Max flow prob

Input: Directed graph G, edge capacities C.

nodes s, t

Def **flow** $f: E \to \mathbb{R}$ st

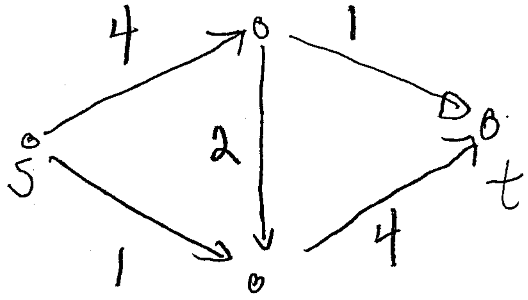
$$0 \le f(u,v) \le C(u,v)$$

$$\sum_{(v,u)\in E} f(v,u) = \sum_{(u,v)\in E} f(u,v) \quad (\text{flow-in} = \text{flow-out})$$

Def Max flow flow f $\sum_{(s,v)\in E} f(s,v)$ maximize.

This is an LP problem!

eg



We did not use skewed symmetric form!

Max Flow Min Cost

Flow Graph G with capacities $c(u,v)$

Cost per edge $a(u,v)$ ie

cost to use $e=(u,v)$ with flow $f(u,v)$ is

$a(u,v) \cdot f(u,v)$

If $a<0$ we get cost back!

Solve using 2 LP's

1) Solve MaxFlow on (G,c) getting flow d

$$\min \sum_{(u,v)\in E} a(u,v)\, f(u,v)$$

subject: $\forall v \neq s,t \quad \sum_{u} f(u,v) = \sum_{u} f(v,u)$ (flow in = flow out)

$$\sum_{v} f(s,v) = d$$

$$0 \le f(u,v) \le c(u,v)$$

Multicommodity Flow

Commodities: $K_1, \ldots, K_K$

Sources: $s_1 \cdots s_K$

Sinks: $t_1, \ldots, t_K$

Variables: $f_i(u,v) \equiv$ flow of commodity K_i on (u,v)

Constraints: $0 \le f_i(u,v) \quad \forall i \in \{1, \ldots, K\} \quad (u,v) \in E$

directed case:

$$\sum_i f_i(u,v) \le c(u,v)$$

$$\forall i \quad \sum_u f_i(u,v) = \sum_u f_i(v,u)$$

undirected case:

$$\sum_{1 \le i \le K} f_i(u,v) + \sum_{1 \le i \le K} f_i(v,u) \le c(u,v)$$

$$\forall i \quad \sum_u f_i(u,v) = \sum_u f_i(v,u)$$

Objective fcns

many choices:

1) fixed demands $d_1, \dots, d_K$

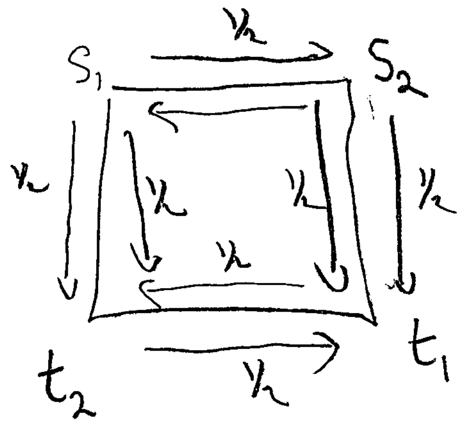
Constraints

$$\sum_v f_i(s_i, v) = d_i \quad i \in \{1, \dots, K\}$$

2) Scaled demands

$$\text{Max} \sum_{i=1}^{K} 1/d_i \sum_v f_i(s_i, v)$$

Note the solution need not be integral!



History

WWII

Kantorovich	1939	
Dantzig	1947	Simplex Alg
von Neumann	1947	Duality
Khachiyan	1979	Ellipsoid Alg
Karmarkar	1984	interior point methods

Smale: 2000

List of 18 greatest unsolved prob in math.

#9 LP prob

#1 Riemann Hypo

#3 $P = NP$