15-750

2/3/16

Basic Depth-First-Search
DFS

Input: $G = (V, E)$ (directed graph)

$v \in V$ (start vertex)

Alg: DFS(G)

1) $\forall u \in V$ color(u) $\leftarrow$ white ; time $\leftarrow 0$

2) $\forall u \in V$ if color(u) $\equiv$ white then DFS-visit(u)

what order?

DFS-visit (u)

1) color(u) $\leftarrow$ gray ; push-time(u) $\leftarrow$ time++

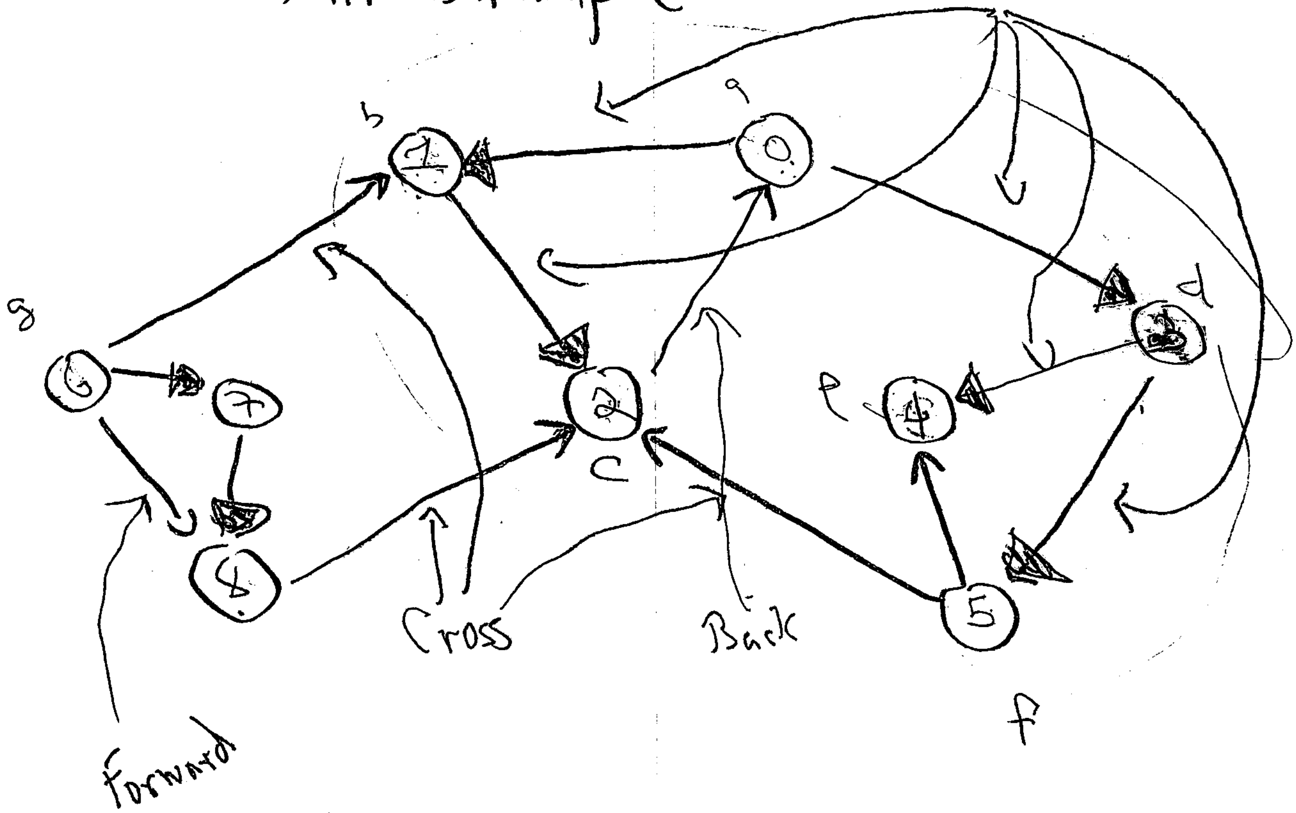
– 2) $\forall v \in$ adj(u)

if color(v) $\equiv$ white then DFS-Visit(v)

3) color(u) $\leftarrow$ black ; pop-time(u) $\leftarrow$ time++

note: dfs(u) $\equiv$ push-time(u)

An Example



Tree

Backedge

Forward

Cross

Testing Edge Types

Conside time that edge $(\overset{e}{\overset{\shortparallel}{u,v}})$ is first used.

Tree(e) iff color(v) = white

BackEdge(e) iff color(v) = gray

color(v) = black iff Cross(e) or Forward(e)

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color(v) = black &	dfs(u) < dfs(v)	forward
"	" > " "	cross

Pop Times

<u>Thm</u> The intervals $[push(u), pop(u)]$ are well nested ie

$push(u) < push(v) < pop(v) < pop(u)$

or $push(u) < pop(u) < push(v) < pop(v)$

(u,v) type edge	pop
Tree	$pop(v) < pop(u)$
back	$pop(v) > pop(u)$
Cross & Forward	$POP(v) < pop(u)$

<u>Thm</u> If G is a DAG & $(u,v) \in E$ then
$pop(v) < pop(u)$

DAG ≡ Directed Acyclic Graph

<u>Thm</u> The following are $\equiv$

a) G has a cycle

b) Every DFS generates a back edge.

c) Some DFS generates a backedge

<u>pf</u> b) $\Rightarrow$ c) $\Rightarrow$ a) Easy

a) $\Rightarrow$ b)

Suppose C is a cycle in G, DFS

Assume that X_1 is first vertex searched

$$C = X_1 \to X_2 \to \cdots \to X_K$$

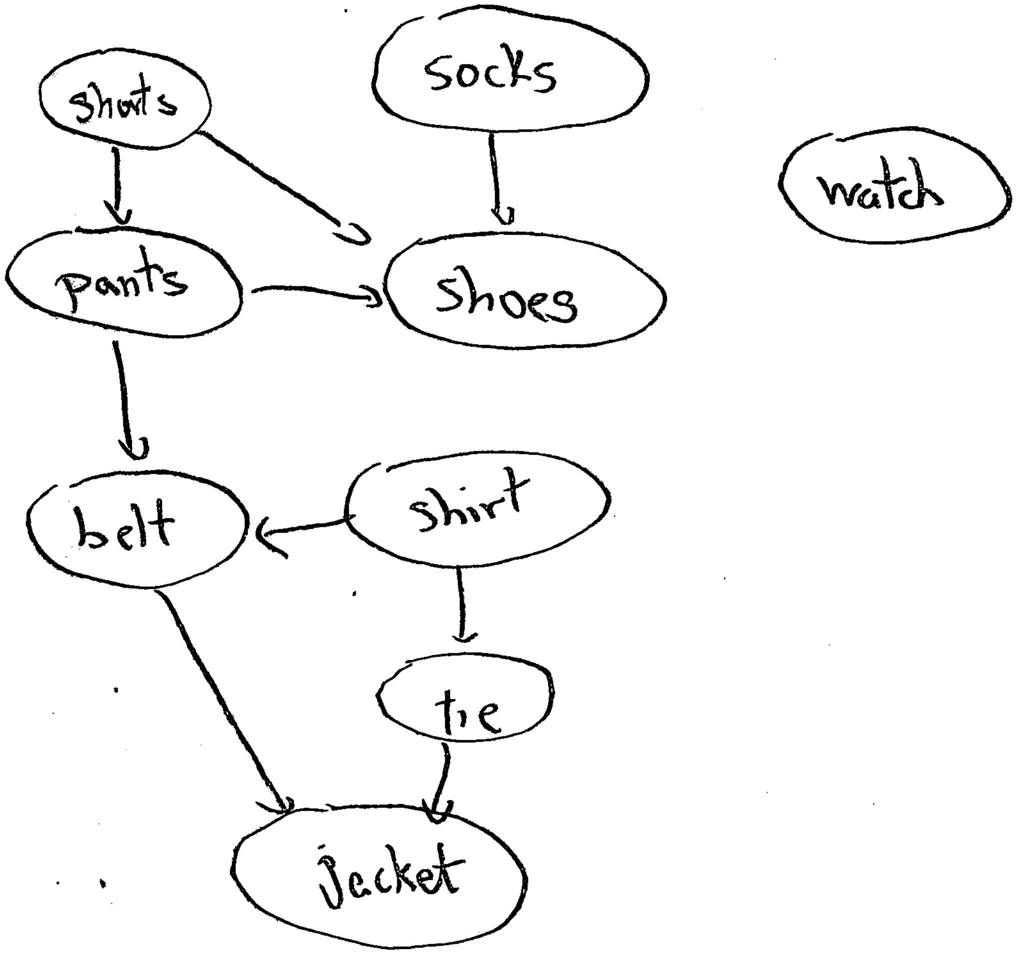
<u>Claim</u> (X_K, X_1) in a backedge

$$push(X_1) < push(X_K) < pop(X_K) < pop(X_1)$$

Topological Sort

<u>Def</u> If $G=(V,E)$ is a DAG then an ordering $x_1 \cdots x_n$ is a <u>topological Sort</u> if

$$(v_i, v_j) \in E \Rightarrow i < j$$



<u>Thm</u> In A DAG reverse pop times is a topological Sort.

ie $a \to b \Rightarrow pop(a) > pop(b)$

<u>Thm</u> Top-Sort is $O(n+m)$ time

Biconnected Components

G is undirected

G is <u>connected</u> if $\forall v, w \in V$ $\exists$ path from v to w.

V is an <u>articulation point</u> if $\exists$ distinct x, y s.t.

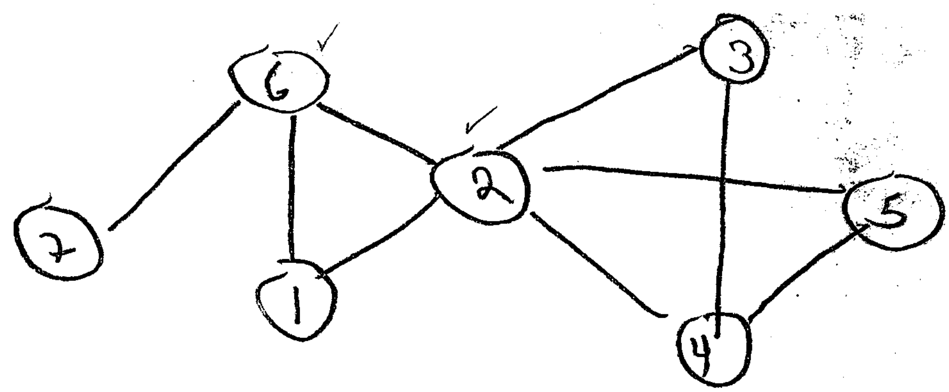
all paths from x to y visit v.

G is connected

<u>Def</u> G is <u>biconnected</u> if $\nexists$ an articulation point

a graph consisting of a single edge is called
a trivial biconnected graph.

<u>Def</u> A <u>biconnected component</u> is a maximal subgraph
which is biconnected

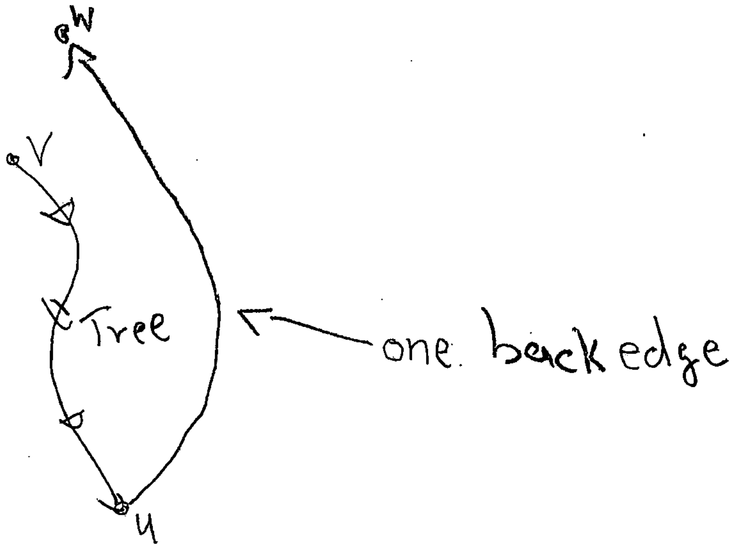


Using DFS for Biconnectivity

<u>Thm</u> In undirected case all edges are tree or backedges

<u>Def</u>

$$low(v) = \min\left\{ \{ DFS(w) \mid \exists u \; u \text{ descendent of } v \land u \to w \text{ back edge} \} \cup \{dfs(v)\} \right\}$$



The Articulation Points after DFS

1) Leaves are not Arts

2) The root is an Art iff # children ≥ 2

3) u is not leaf & not a root then

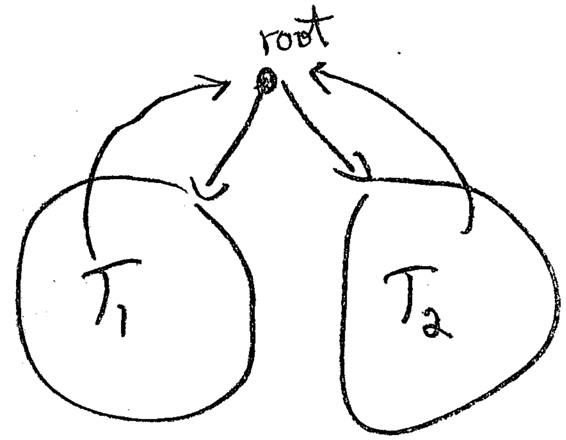
u is an Art iff $\exists$ child v s.t. $low(v) \geq dfs(u)$

Pf

1) If v is a leaf then $T - \{v\}$ is connected

2) If root has 1 child then $T - \{root\}$ is connected.
≥ 2 children " " not connected

Any path from one child to other uses root



pf

3) $(\Rightarrow)$ suppose paths from x to y use u

3a) $x, y \in \text{Subtree}(u) \wedge$ suppose $low(x), low(y) < dfs(u)$

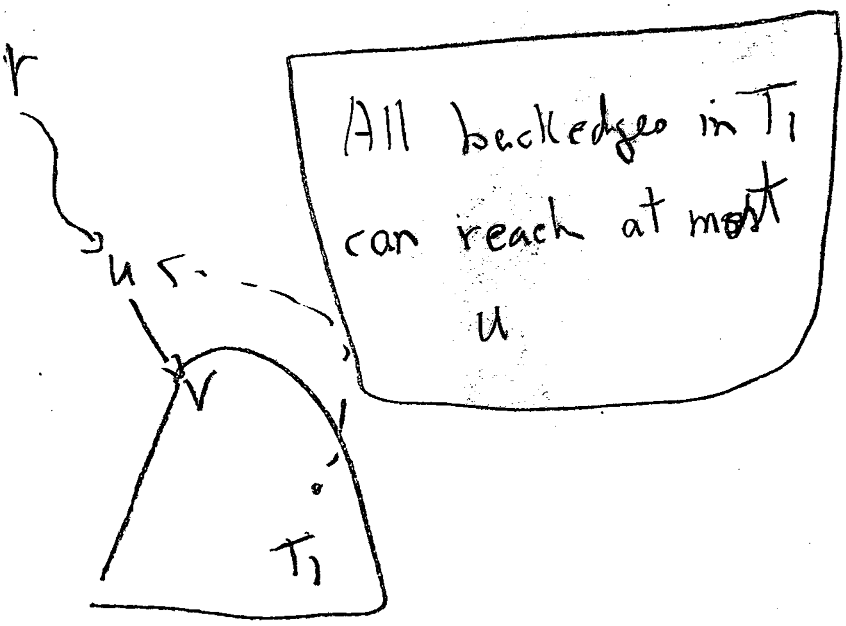
then $\exists$ path from x to y not using u. contra!

3b) $x \notin \text{subtree}(u)$ then

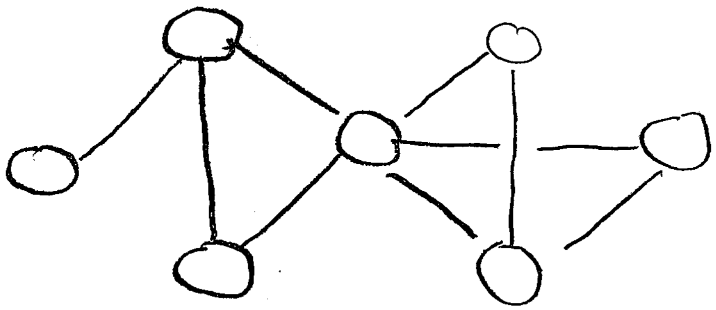
$low(y) \geq dfs(u)$

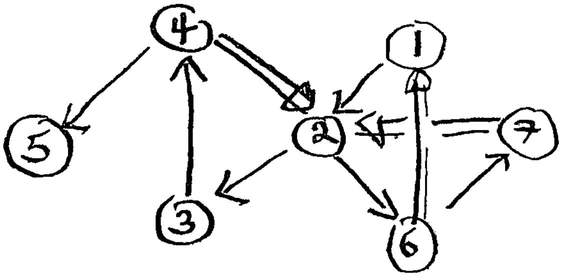
$(\Leftarrow)$ $v = child(u)$ $low(v) \geq dfs(u)$

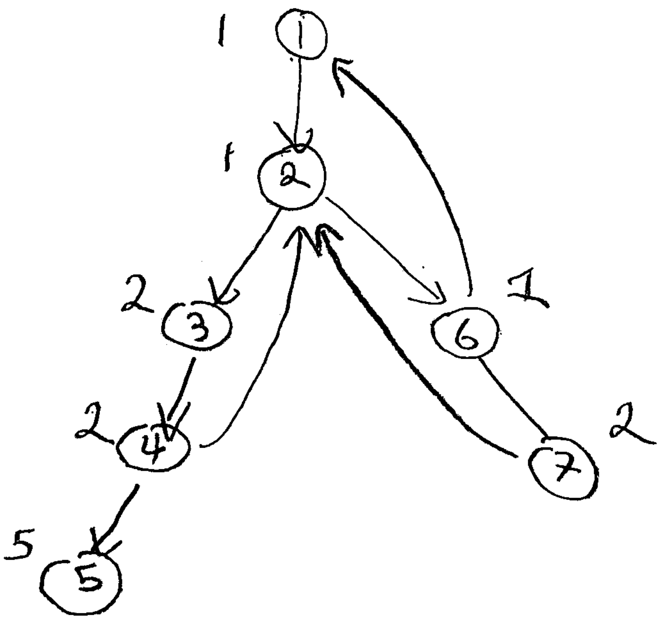
u separates v from r.



Example







Arts 2, 4

Computing low(u)

1) $\forall u$ $low(u) \leftarrow dfs(u)$

if (u,v) is back edge

$low(u) \leftarrow \min\{low(u), dfs(v)\}$

if (u,v) tree edge

$low(u) \leftarrow \min\{low(u), low(v)\}$