15-750
4/14/10

Graph Separators

Let $G = (V, E)$ undirected

In general: vetices & edges have weight

$\# : V \cup E \to R^+$ $\quad \sum_{v \in V} \#(v) + \sum_{e \in E} \#(e) = 1$

<u>Vertex Separator</u> $S \subseteq V$ is an <u>$f(n)$-separator</u> if:

1) $|S| \leq f(n)$ $\quad n = |V|$ (small)

2) $\exists$ partition A, B of $V - S$ (balanced) s.t. $|A|, |B| \leq \frac{2}{3} n$

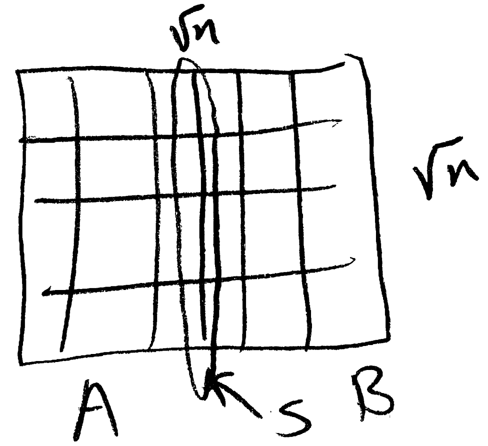
3) No edge from A to B (separates)

<u>Def</u> $\#(A) = \sum_{v \in A} \#(v) + \sum_{(v,w) \in A} \#(v,w)$

2') $\#(A), \#(B) \leq 2/3$

eg Let $M_n \equiv \sqrt{n} \times \sqrt{n}$ mesh graph

<u>Claim</u> M_n has a $\sqrt{n}$-separator



Edge Separators

$S \subseteq E$ is a <u>$f(n)$-separator</u>

1) $|S| \leq f(n)$

2) $\exists$ partition A, B of V $|A|, |B| \leq \frac{2}{3}n$

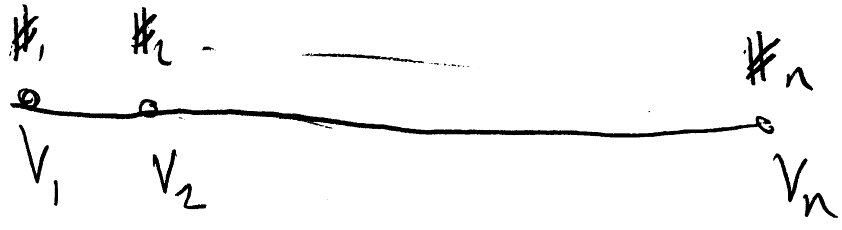
3) no edge in $E - S$ from A to B

Weighted edge separators

2) = $\exists$ part A, B of V $\#(A), \#(B) \leq \frac{2}{3}$

<u>Claim</u> Vertex weighted line graph has an edge 1-separator if $\forall v \in V \ \#(v) \le 1/3$

<u>pf</u>



$\sum w_i = 1$

Let $S_j = \sum_{i \le j} \#_i$

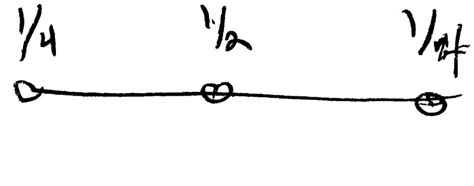
pick first j st $S_j > 1/3$

<u>Claim</u> $e_{j,j+1}$ works ie $S_j \le 2/3$ & $(1-S_j) \le 2/3$

a) $S_{j-1} \le 1/3$ & $\#_j \le 1/3 \Rightarrow S_j \le 2/3$

b) $S_j > 1/3 \Rightarrow 1 - S_j < 2/3$

What if some $\#_i > 1/3$?



<u>Claim</u> Vertex & edge weighted tree has a vertex 1-separator if $\forall e \in E \ \#(e) \leq 1/3$

<u>pf</u> If $\exists v \in V$ st $\#(v) > 1/3$ pick v.

Direct each edge e of T as follows:

$T \setminus e$ gives 2 trees T_1 T_2

if $\#(T_1) \leq \#(T_2)$ direct from T_1 to T_2

(break ties)

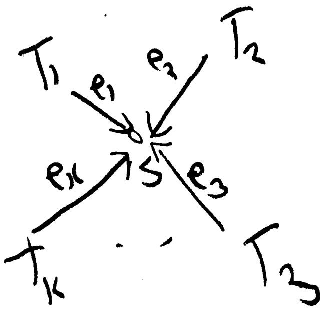
<u>Claim</u> $T_1 \xrightarrow{e} T_2$ then $\#(T_1) + \#(e) \leq 2/3$

ie $\#(T_2) \geq 1/3$

Suppose $\#(T_2) < 1/3$ then $\#(T_1) + \#(T_2) + \#(e) < 1$

contra!

Since T is a tree there must $\exists$ a sink $s \in V$.



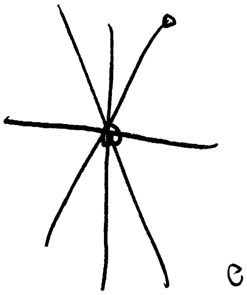
Let $\#_i = \#(T_i) + \#(e_i)$

We know that each $\#_i \leq 2/3$

Thus there must exist a partition A, B st $\#(A), \#(B) \leq 2/3$ Why?.

Claim: A vertex & edge weighted tree of degree at most 3 has an edge 1-separator if $\forall v \in V$ $\#(v) \leq 1/3$.

Do all trees have small edge separator?

$S_n \equiv$ 

<u>Thm</u> (Planar Separator Thm)
Every planar graph has a vertex $\sqrt{8n}$-separator.
Know bounds: $\sqrt{6n}$!

Assume we have an embedded & triangulated planar graph G.

<u>We prove $4\sqrt{n}$-separator</u>

BFS step of Planar Separator Thm

$G \equiv$ embedded tri planar graph

1) Pick some $s \in V$
2) Using BFS level the vertices of V.

   i.e. $\text{level}(s) = 0$

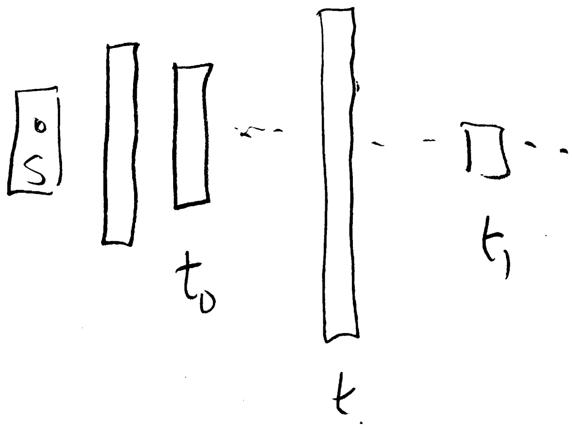
   Let $L(t) \equiv$ set of vertices at level t.

   Let $A_t = \bigcup_{i<t} L(i) \quad S_t = L(t) \quad B_t = \bigcup_{i>t} L(t)$

3) Pick t st $|A_t|, |B_t| \leq n/2$
4) If $|S_t| \leq 4\sqrt{n}$ done
5) Set $t_0 = \max_{t'<t} |L(t')| \leq \sqrt{n}$

   $t_1 = \min_{t'>t} |L(t')| \leq \sqrt{n}$

6) Remove $L(t_0)$ & $L(t_1)$ and their edges



$|L(t')| > \sqrt{n}$ for $t_0 < t' < t_1$

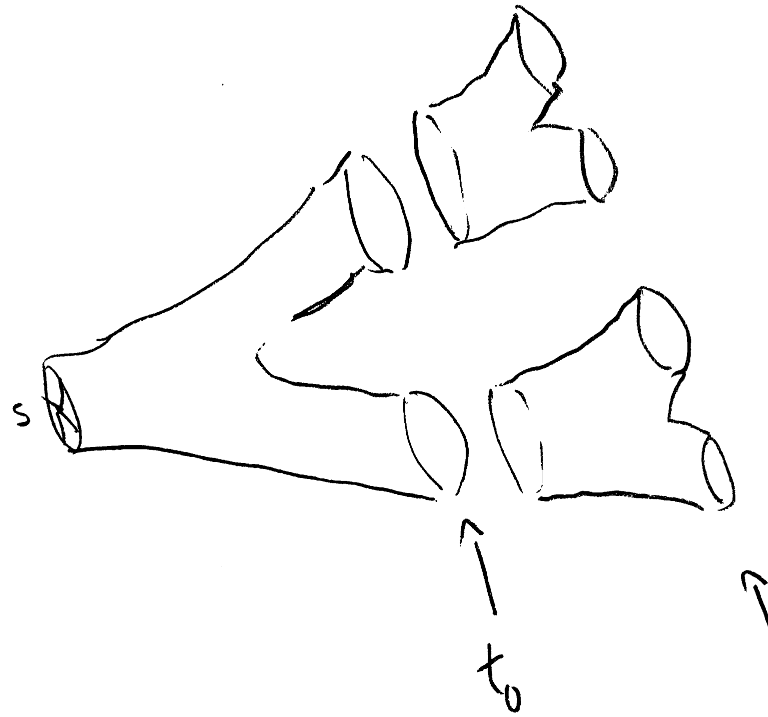
so $(t_1 - t_0 - 1)(\sqrt{n}+1) \leq n-1$ $\quad |L(t_0)| \geq 1$

$(t_1 - t_0)(\sqrt{n}+1) \leq n-1+\sqrt{n}+1 = n+\sqrt{n} = \sqrt{n}(\sqrt{n}+1)$

$(t_1 - t_0) \leq \sqrt{n}$

<u>Thus</u> $(t_1 - t_0) \leq \sqrt{n}$

Planar graph on bark of a tree



After removing $L(t_0)$ & $L(t_1)$

7) Find connected components

8) If size of each component $\le \frac{2}{3}n$ done.
else C be a component $|C| > \frac{2}{3}n$

9) Triangulate open (new) faces of C.

10) Find spanning tree T of C with
diameter $\le 2\sqrt{n}$

Trees in Planar Graphs

Let $G=(V,E)$ be an embedded planar graph
Let $G^*=(V^*,E)$ be its dual.

Lemma Let $E' \subseteq E$
(V,E') has a cycle iff $(V^*, E-E')$ is disconnected

$\Rightarrow$ Suppose C is a cycle in (V,E') then no face "inside" C can reach a face "outside" of C using only edges in $E-E'$

$\Leftarrow$ Suppose $H=(V^*, E'')$ is a maximal disconnected subgraph of G^*. then the boundary between the 2 connected components of H must be a cycle in (V,E).

<u>Lemma</u> $T=(V,E')$ is a spanning tree of G

iff $T^*=(V^*,E-E')$ " G^*.

Pf Suppose T is a spanning tree then

T is connected & cycle free

Thus T^* is cycle free & connected

$\therefore T^*$ is a Tree

Since $(G^*)^* = G$ we are done!

Suppose T is a spanning tree & $e \notin T$

$\exists!$ cycle formed from e & T. C_e,

the <u>induced</u> cycle.

<u>Thm</u> Let G be a tri planar graph & T a

Spanning tree then some induced cycle

is a separator.

<u>Pf</u> 2) Thus, "if the goal was to separate faces we

would be done." 1) Since T^* is a degree 3 tree.

We know that degree 3 trees have an

edge separator.

Let $e \notin T$ assume that e is oriented

$In(e) \equiv$ weight of interior of C_e

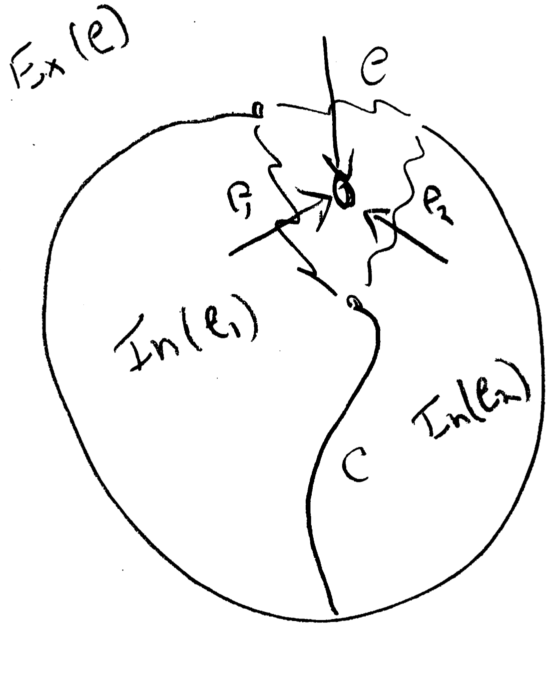
$Ex(e) \equiv$ weight of exterior of C_e

If $In(e) < Ex(e)$ then direct e^* from Interior to Exterior.

If $Ex(e) < In(e)$ " " Ext to Int

If $Ex(e) = In(e)$ done

Suppose face F is a sink



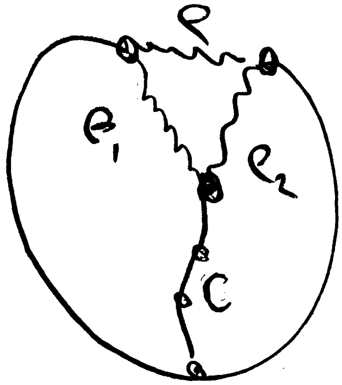
Concern:

$In(e) > \frac{2}{3}n$ &

$Ex(e_1), Ex(e_2) > \frac{2}{3}n$.

<u>Claim</u> 1) $In(e) > \frac{2}{3}n$ & 2) $In(e_1) \geq In(e_2)$

then $Ex(e_1) < \frac{2}{3}n$



pf

$In(e) = In(e_1) + In(e_2) + \#(C)$

$In(e_1) + In(e_2) + \#(C) > \frac{2}{3}n$ by 1)

$2\, In(e_1) + \#(C) > 2/3\, n$ by 2)

$In(e_1) + \frac{\#(C)}{2} > 1/3\, n$ divide by 2

$In(e_1) + \#(C) > 1/3\, n$ $\#(C) \geq 0$

$Ex(e_1) < 2/3\, n$ $\sum \# = n$

Thus $V(C_{e_1})$ is a separator of component C from step C.

Final Planar separator

1) $L(t_0) + L(T_1) + C_e$

$$\sqrt{n} + \sqrt{n} + 2\sqrt{n} \leq 4\sqrt{n}$$