Resistive Theory

Network (Resistive)
undirected graph $G=(V,E)$ (oriented)
conductance $C: E \to \mathbb{R}^+$

Flow (Kirchhoff flow) from a to b.

$j: E \to \mathbb{R}$ s.t.

1) $j_{xy} = -j_{yx}$

2) $\sum_y j_{xy} = 0$ if $x \neq a, b$

3) $j_{xy} = 0$ if $(x,y) \notin E$

Def i is an Ohms flow if

$\exists v: V \to \mathbb{R}$ st

$i_{xy} = C_{xy}(V_x - V_y)$ $(x,y) \in E$

Def Residual flow $i_x = \sum_y i_{xy}$

Graph Laplacians

$$L_G = L = D - A \qquad D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \qquad A = \text{Adj}$$

$$\Gamma = \begin{array}{c|c} & v_1 \cdots v_n \\ \hline e_1 & \\ \vdots & \boxed{} \\ e_m & \end{array} \qquad L = \Gamma^T C \Gamma \qquad C = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_m \end{pmatrix}$$

<u>Thm</u> If $V = \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix}$ voltages with residual currents $i = \begin{pmatrix} i_1 \\ \vdots \\ i_n \end{pmatrix}$

then $LV = i$

Energy / Power

<u>If</u> $V \equiv$ voltages then $V^T L V = \sum c_{xy} (V_x - V_y)^2 \equiv$ Energy

<u>Note</u> $LV = 0$ & G connected then $V = \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix}$

Rank $V = n-1$

Suppose $a, b \in V$ & effective resistance R_{ab}

Def Effective energy $i_{ab}^2 R_{ab} = R_{ab}$ if $i_{ab} = 1$

Real energy is:

solve $LV = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$ $V_a = V_1$ & $V_b = V_n$

Energy $V^T L V = V^T \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} = V_a - V_b$

Thm Real energy = effective energy

Rayleigh's Monotonicity Law

Network $G = (V, E, C)$ $\bar{G} = (V, \bar{E}, \bar{C})$

If $\forall x, y \in V$ $\bar{R}_{xy} \geq R_{xy}$ then $\bar{ER}_{ab} \geq ER_{ab}$

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pf need two Thm's first.

Let $j \equiv$ unit Ohms flow from a to b in $\bar{G}$

i " G

$$
\begin{aligned}
\bar{ER}_{ab} &= 1^2\, \bar{ER}_{ab} && \text{effective energy} \\
&= \tfrac{1}{2} \sum j_{xy}^2 \bar{R}_{xy} && \text{reod energy} \\
&\geq \tfrac{1}{2} \sum j_{xy}^2 R_{xy} && \text{Hypothesis} \\
&\geq \tfrac{1}{2} \sum i_{xy}^2 R_{xy} && \text{(Thomson)} \\
&= ER_{ab} && \text{effective}
\end{aligned}
$$

Conservation of Energy

Let $W \equiv$ any voltage settings
$j \equiv$ any flow from a to b.

<u>Thm</u> $$(W_a - W_b)\, j_a = \tfrac{1}{2} \sum_{x,y} (W_x - W_y)\, j_{xy}$$

$$2\,\text{RHS} = \sum_{x,y} (W_x - W_y)\, j_{xy} = \sum_x W_x \sum_y j_{xy} - \sum_y W_y \sum_x j_{xy}$$

$$= W_a \sum_y j_{ay} + W_b \sum_y j_{by} - \left(W_a \sum_x j_{xa} + W_b \sum_x j_{xb} \right)$$

$$= W_a j_a + W_b j_b - W_a(-j_a) - W_b(-j_b)$$

$$= W_a j_a - W_b j_a + W_a j_a - W_b j_a$$

$$= 2(W_a - W_b)\, j_a$$

Thomson's Principle

i & j = unit flows from a to b

i satisfies Ohm's Law then

$$\sum i_{xy}^2 R_{xy} \leq \sum j_{xy}^2 R_{xy}$$

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pf let $d = j - i$ then d is a zero flow from a to b.

$$\sum j_{xy}^2 R_{xy} = \sum (i_{xy} + d_{xy})^2 R_{xy}$$

$$= \sum i_{xy}^2 R_{xy} + 2\sum i_{xy} R_{xy} d_{xy} + \sum d_{xy}^2 R_{xy}$$

$$+ 2\sum (V_x - V_y) d_{xy} \quad (*)$$

Claim $(*) = 4(V_a - V_b) d_a = 0$ by

conservation of energy.

<u>thus</u> $$\sum j_{xy}^2 R_{xy} = \sum i_{xy}^2 R_{xy} + \sum d_{xy}^2 R_{xy}$$

$$\geq \sum i_{xy}^2 R_{xy}$$