15-750
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The Preflow-Push Max-Flow Alg

Goal: $O(n^3)$ Alg $O(n^2 m)$

This Alg + Better Data Structures: $O(n \cdot m \log n)$

Known: $O(m\sqrt{n} \log^k n)$ (some k)

Idea Relax the flow-in = flow-out condition.

Def f is an s-t preflow (preflow) if

1) $f: E \to R^+$
2) $0 \le f(e) \le c_e$
3) $e_f(v) = \sum_u f(u,v) - \sum_u f(v,u) \ge 0$ (excess)

$\forall v \in V \setminus \{s,t\}$

As before we have a residual graph.

G_f

We use a vertex label to guide "pushing" the flow.

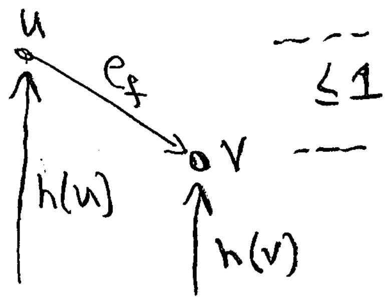
Let $h: V \to \mathbb{Z}^+ = \{0, 1, 2, \cdots\}$ (label or height)

<u>Def</u> f & h are <u>compatible</u> if

1) $h(t) = 0$ & $h(s) = n$ (sink & source condition)

2) $\forall (u,v) \in E_f \quad h(u) \leq h(v) + 1$

eg



<u>Lemma</u> f (st preflow) & h (compatible labeling)

then no augmenting path in G_f.

<u>pf</u> By contradiction! Let P be such a path

We may assume P is simple. # edges of P < n

$\therefore$ $h(s) - h(t) < n$ contra!

<u>Cor</u> f (s-t flow) & h (compatible labeling)

then f is a Max-Flow.

Init(G) $\quad f(s,v) = C(s,v) \quad \forall v$

$f(e) = 0 \quad$ o.w.

$h(s) = n \quad h(v) = 0 \quad$ o.w.

<u>Note</u> f & h are compatible

<u>pf</u> Only v st $h(v) > 0$ is s.

But we saturated edges out of s.

Note f is only a preflow!

Push & Relabel

Push (f, h, v, w)

if $e_f(v) > 0$ & $(v,w) \in E_f$ & $h(v) > h(w)$

then "push" $\min\{e_f(v); c_f\}$

eg

$e_f(v) = 1/2$

$h(v) = 10$ & $h(w) = 9$

Relabel (f, h, v)

If $e_f(v) > 0$ & $\forall (v,w) \in E_f$ $h(v) \le h(w)$

then $h(v) \leftarrow h(v) + 1$

Preflow-Push

1) Init
2) While $\exists\, v \neq t$ st $e_f(v) > 0$ do
   If $\exists\, w$ st $(v, w) \in E_f$ & $h(v) > h(w)$
   then Push(f, h, v, w)
   Else relabel(f, h, v)
3) Return f

Correctness

Claim 1 f & h are always compatible.

Claim 2 If Preflow-Push returns f then f is a max-flow

pf of claim 1

Each push preserves compatiblity.
" " relabel " ".

Timing & Termination

To show excess can be "pushed" back to s.

Lemma f preflow & $e_f(v) > 0$ then $\exists$ path from v to s. (s not t)

pf Let $A = \{ w \mid \exists \text{path from } w \text{ to } s \text{ in } G_f \}$

$B = V - A$

Note Flow f can only go from B to A.

Claim $\sum_{v \in B} e_f(v) = 0$

$$\sum_{v \in B} e_f(v) = \sum_{v \in B} f^{in}(v) - f^{out}(v)$$

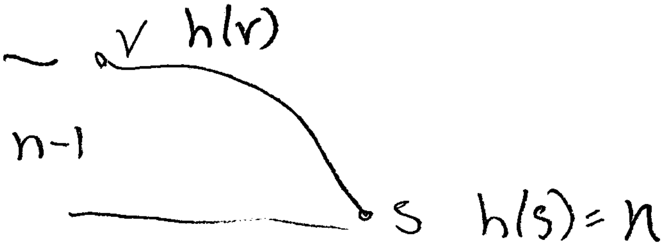
three types of edges $(u,v) \in$ sum

1) $u, v \in B$ (cancel)
2) $u \in A$ & $v \in B$ (zero)
3) $v \in B$ $u \in A$ (negative in sum)

i.e. $0 \le \sum_{v \in B} e_f(v) = -f^{out}(B) \le 0$ **QED**

Claim $h(v) \le 2n-1$

pf



$\Rightarrow h(v) \ge 2n-1$

Cor # of relabels $< 2n^2$

Counting # of pushes

2 types

saturating push $= c_f$

nonsaturating otherwise

Claim # saturating pushes $\le 2nm$

to show # per edge $\le n$

Before push $\; v \xrightarrow{e_f} w \quad h(v) = h(w)+1$

After push $\; v \xleftarrow{e_f} w \quad$ need 2 relabels.

Lemma # nonsaturating pushes $\le 4n^2m$.

pf Consider the following potential

$$\Phi(f,h) = \sum_{e_f(v)>0} h(v)$$

ops	#	AC	unit cost	effect on potential
relabel	$2n^2$	≤ 2	1	$+2n^2$
saturating	$2nm$	$\le 2n-1$	$\le 2n-1$	$+2nm(2n-1)$
non-sat	α	0	1	$-\alpha$

$$\Rightarrow \alpha \le 2nm(2n-1) + 2n^2 = 4n^2m - 2nm + 2n^2 \le 4n^2m$$

We have shown

<u>Thm</u> Preflow-Push is $O(n^2 m)$ time.

<u>New Alg</u> Preflow-Push-Max-Height

<u>Idea:</u> Only "push" max height

Let $H = \max_{e_f(v)>0} h(v)$

<u>Push</u> (f, h, v, w)

<u>if</u> 1) $e_f(v) > 0$
2) $(v,w) \in E_f$
3) $H = h(v) > h(w)$

<u>then</u> "push" $\min\{e_f(v), c_f\}$

<u>Thm</u> Preflow-Push-Max-Height in $O(n^3)$ time.

Potential: $\Phi(f,h) = \max\{h(v) : e_f(v) > 0\}$

<u>Note:</u> Only relabels increase Φ.

$\#$relabel $\leq 2n^2$

$\#$decreases in $\Phi \leq \#$ increases

Thus $\leq 4n^2$ total changes.

Counting the number of pushes:

Saturating: $\#SAT \le 2nm \le n^3$

Non-Sat: Relabel only if cannot find a push.

<u>Note</u> push(v,w) non-sat then $e_f(v) = 0$

$\therefore \left| \{v \mid e_f(v) > 0 \ \& \ h(v) = H\} \right|$

is one less.

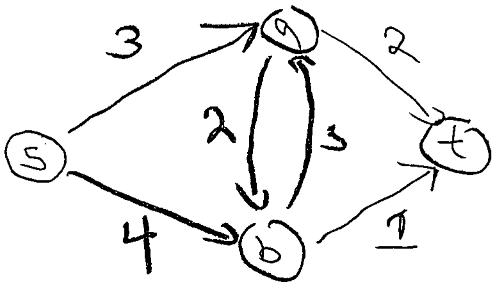
Goal: # of non-sat pushes per change in $\Phi(f,h)$

$\le n$

$\therefore$ # non-sat pushes $\le 4n^3$

QED.

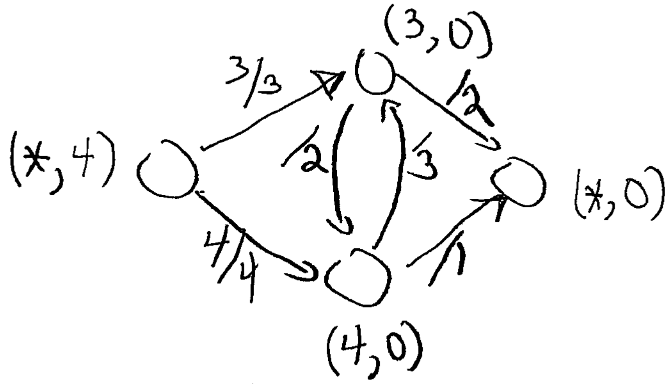
Flow Prob.



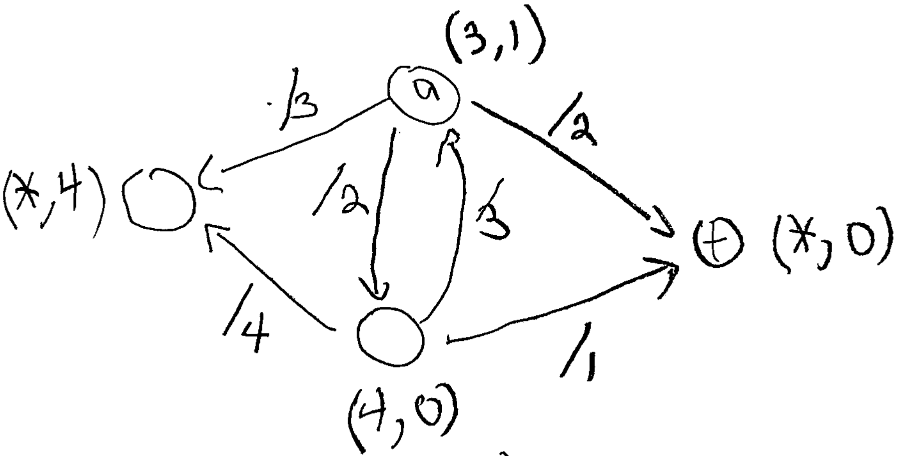
excess

(3,0) height

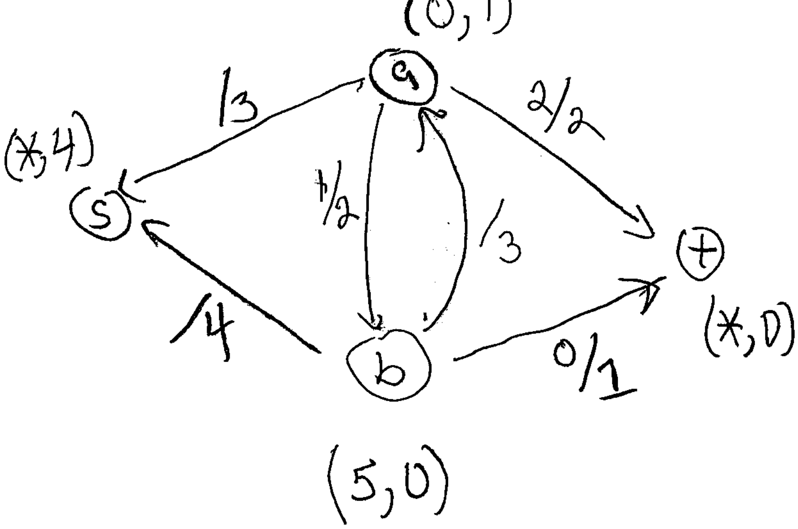
Init



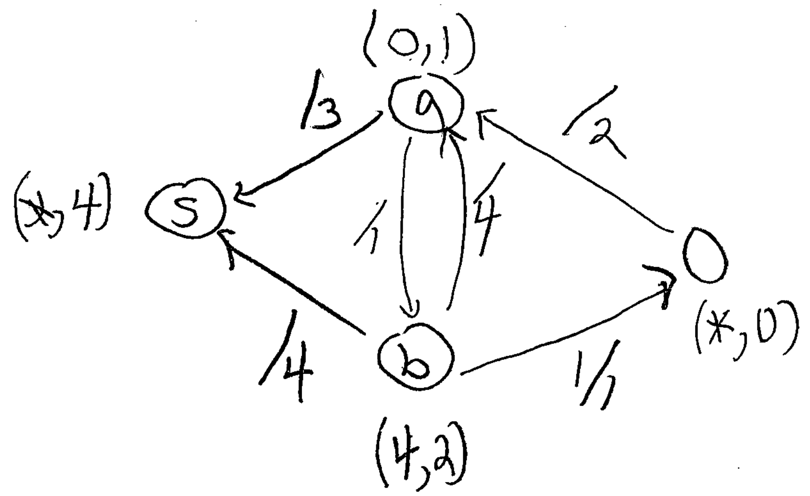
1) residual
2) relabel (a)



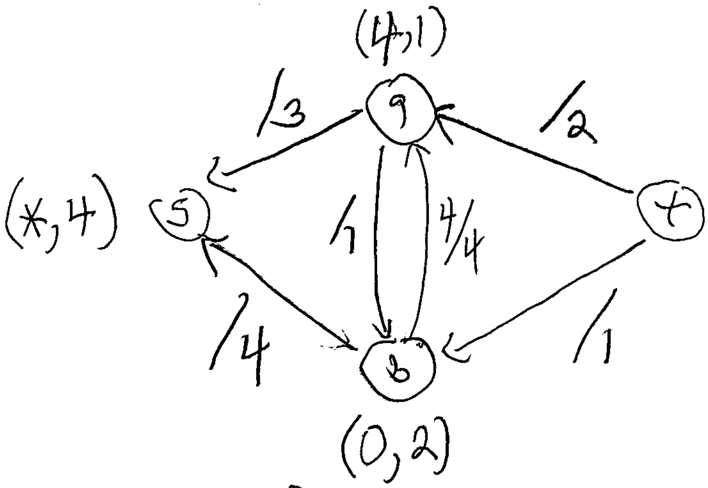
1) Push(f, h, a, t)
Push(f, h, a, b)



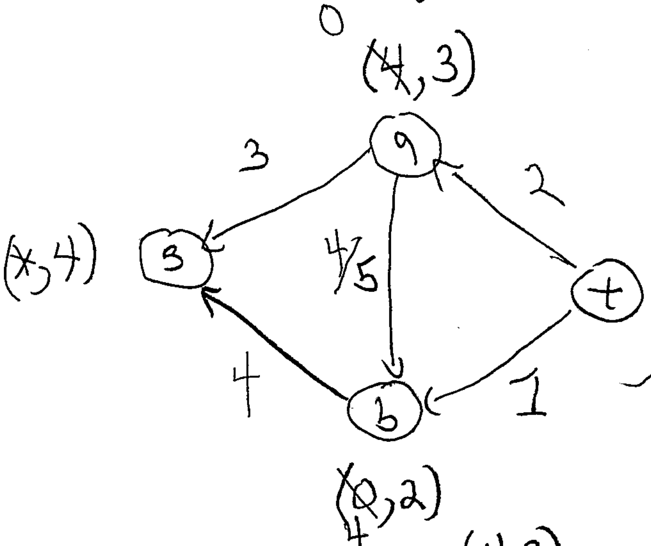
B



1) residual
2) relabel(b)
3) push(b,t)
4) relabel(b)

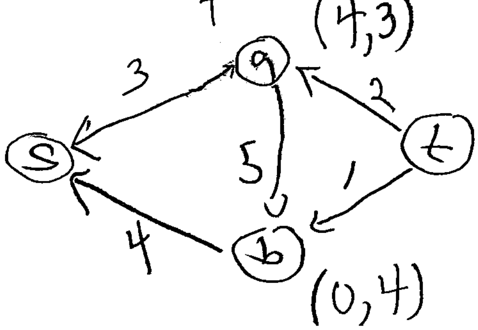


1) push(b,a)

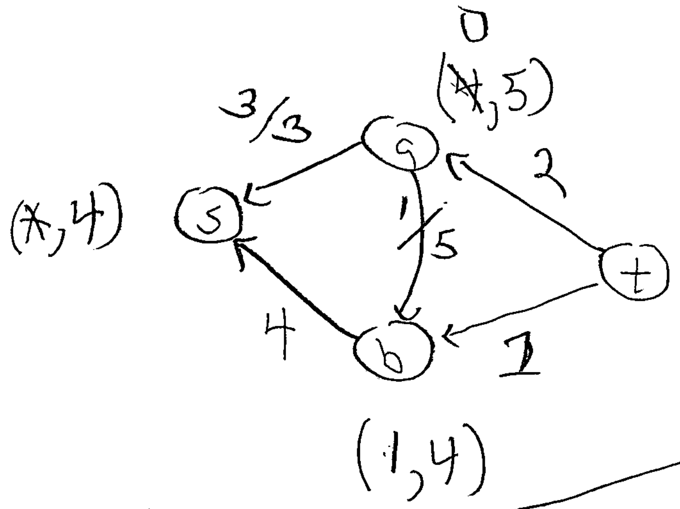


1) relabel^2(a)
2) push(a,b)

3) relabel^2(b)
4) push(b,a)



C



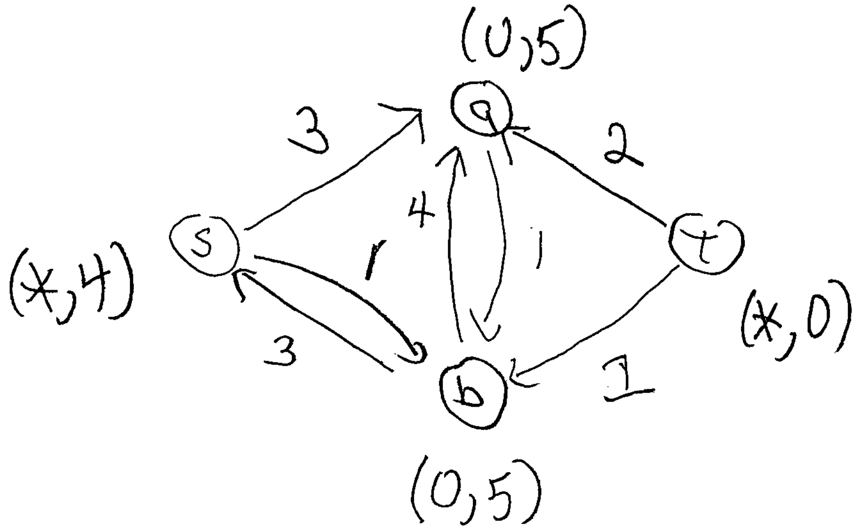
1) relabel^2(a)

2) push(a, s)

3) push(a,b)

4) relabel(b)

5) push(b,s)



final Flow