Parallel Algorithms III

15-750
4/4/16

Topics

1) Randomized List-Ranking

A Linked List

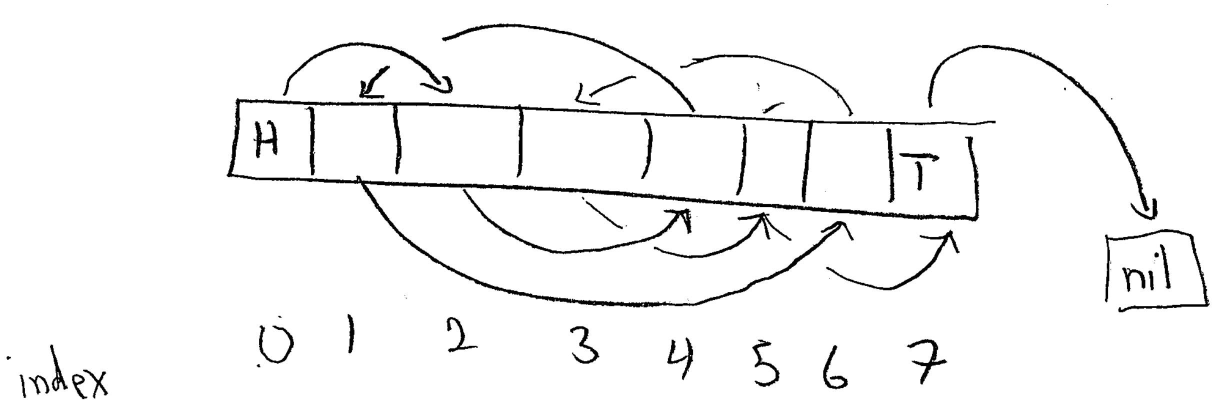
<u>Def</u> Directed graph with head & tail

1) indegree = 1 except indegree(head) = 0
2) outdegree = 1 except outdeg(tail) = 0
3) Connected

Further Properties needed for parallell Alg.

1) Pointers in consecutive memory
1') Each pointer has an index which we can access in unit time.

eg



Random-Mate (Randomized Alg)

1) Contraction Phase

1) Each live node randomly picks a sex.

2) If $F \xrightarrow{a} M \xrightarrow{b} X$ then

$$\begin{tikzcd} F \arrow[rr, bend left, "a+b"] & M & X \end{tikzcd}$$

dies

3) Stop when head points to nil.
(only head is live)

<u>Thm</u> The contraction phase stops in $c \log n$ rounds with high prob.

<u>pf</u> Let $P_i \equiv$ Event that node i is still live after one round

<u>Note</u> node_i not head then $\text{Prob}[P_i] = 3/4$

Let $P_i^k \equiv$ Event that node i still live after k rounds.

Note; $\text{Prob}[P_i^k] = (3/4)^k$ i not head. (By independence)

set $k = \lfloor c \log_{4/3} n \rfloor$

$$\text{Prob}[P_i^k] = 1/(4/3)^k \le \frac{1}{(4/3)^{c \log_{4/3} n}} = \frac{1}{n^c}$$

Let $P^k \equiv$ Event that some non-head node is still live.

assume that $node_0$ is the head.

$$P^k = P_1^k \cup P_2^k \cup \cdots \cup P_n^k$$

$$\text{Prob}[P^k] = \text{Prob}[P_1^k \cup \cdots \cup P_n^k]$$

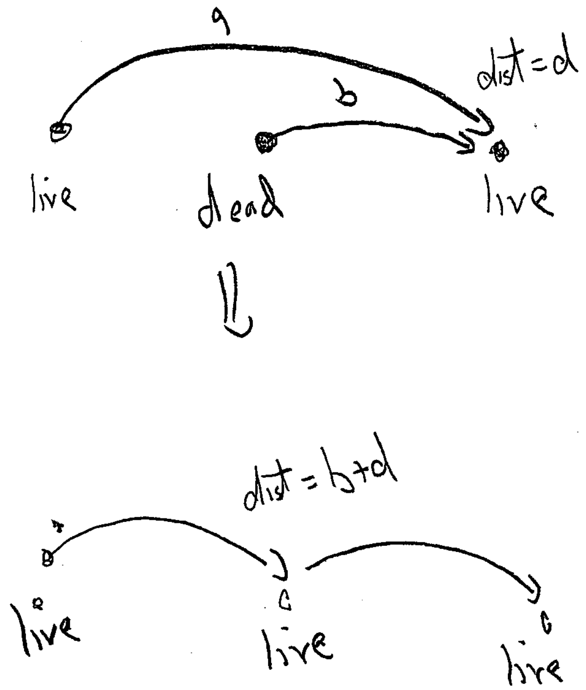
$$\leq \text{Prob}[P_1^k] + \cdots + \text{Prob}[P_n^k]$$

$$\leq n \cdot \frac{1}{n^c} = \frac{1}{n^{c-1}}$$

If we set $c=2$ then the contraction phase stops with prob $\leq 1/n$ in $2\log_{4/3} n$ rounds.

In the expansion phase we run contraction phase "backwards".

ie



The same as "down" in Prescan.

Note: P.T $= O(n \log n)$ with high prob.

Goal: An Alg P.T $= O(n)$ with HP.

A Simple Randomized List-Ranking

Assume Linked-list is doubly linked.

Alg Splicing-out

1) Make $n/\log n$ queues of size $\log n$ (queue/proc)

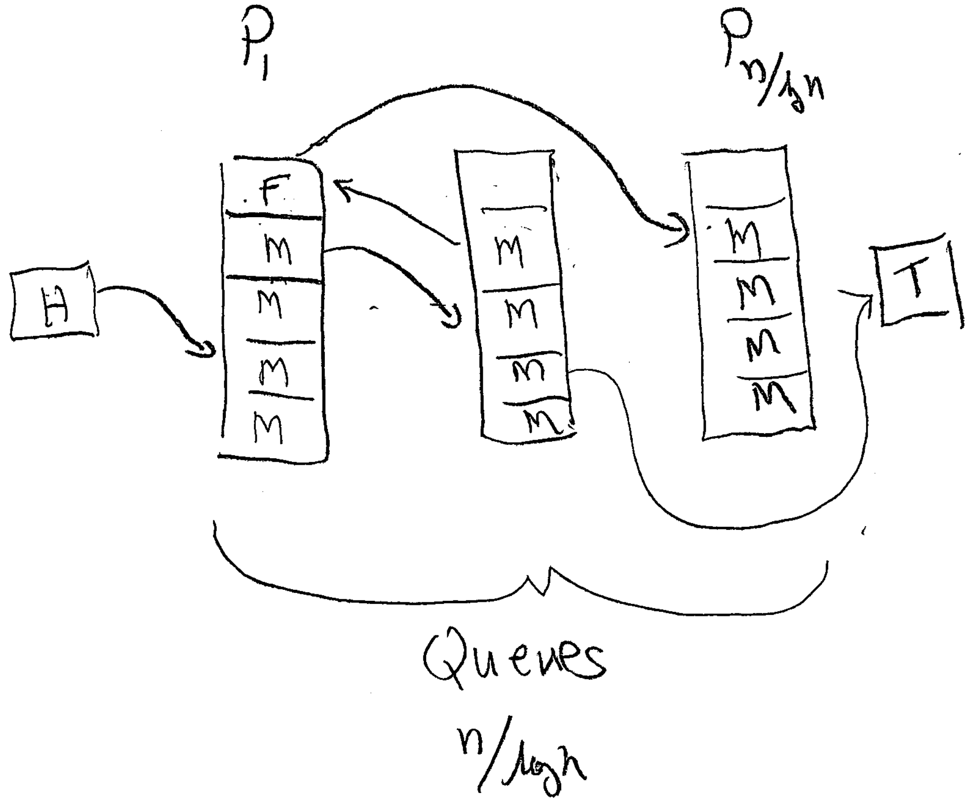
2) Set sex of all nodes to M.

3) Reset sex of each queue-top to random sex.

4) In ll If top is F and points to M then "splice-out" top.

5) Repeat 3)&4) while some queue not empty.

Thm After $O(\log n)$ rounds all queues are empty with high probability.



Queue size $\log n$

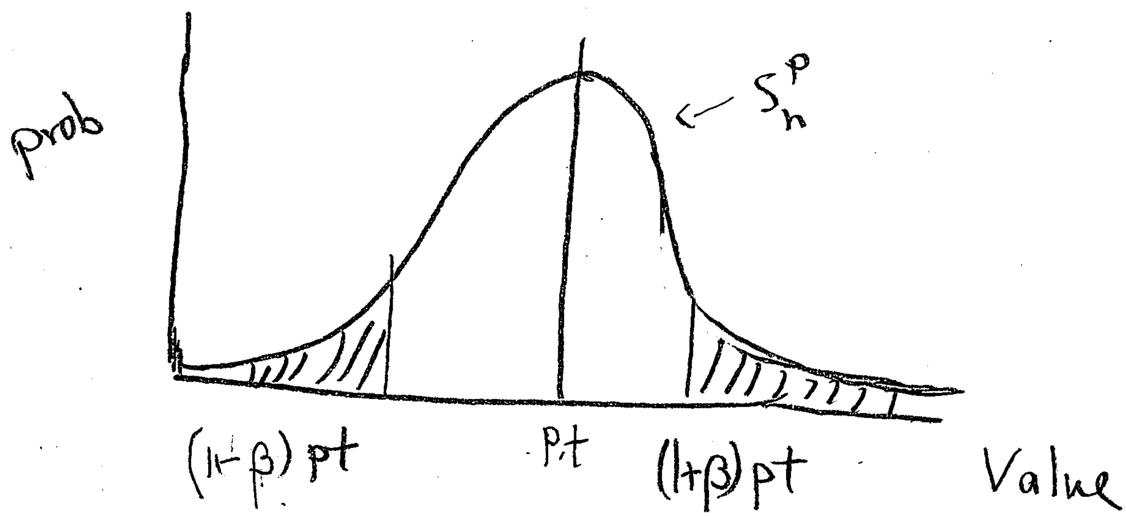
Chernoff Bounds

Let $X_1 \cdots X_t$ be independent 0/1 random variables

Assume $\mathrm{Prob}(X_i = 1) = p$

The binomial random variable is

$$S_n^p = X_1 + \cdots + X_t$$



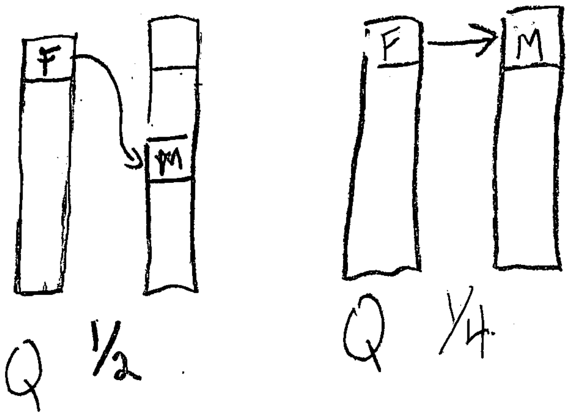
$$\mathrm{Expect}(S_t^p) = \sum E(X_i) = p \cdot t$$

$\underline{\text{Thm}}$ $\text{Prob}\left(S_t^p < (1-\beta)pt\right) < e^{-\beta^2 pt/2}$ $\forall\, 0 \leq \beta < 1$

$\underline{\text{Thm}}$ $\text{Prob}\left(S_t^p > (1+\beta)pt\right) < e^{-\beta^2 pt/3}$ $\forall\, 0 \leq \beta < 1$

Lets fix one of the queues, say, Q.

At a given round the prob Top is spliced-out is $\geq 1/4$



View prob as:

We have a coin Prob (Head) = $1/4$ Prob(Tail) = $3/4$

Question: After t flips what is

$$\text{Prob}[\#\text{heads} < \log n] ?$$

Suppose we pick t s.t. Expect # heads $= 4 \log n$

ie $t = 16 \log n$

Here: $\log \equiv$ log base 2. Could use base e

$t = 11 \log_2 n$

We apply Chernoff with

$p = 1/4$, $t = 16 \log n$ $\beta = 3/4$

$$\text{Prob}\left(S_t^p < (1-\beta)pt\right) < e^{-\beta^2 pt/2}$$

$$\text{Prob}\left(S_t^p < \log n\right) < e^{-(3/4)^2(1/4)\,16(\log n)(1/2)}$$

$$= e^{-9/8 \log n} \leq n^{-9/8}$$

Thus Prob that some queue is not empty after $t = 16 \log n$ rounds $< (n/\log n)\, n^{-9/8}$

$$< n^{-1/8}$$