15-750 1/11/16

Graduate Algorithms

Instructor: Gary Miller

TA's

Vijay Bhattiprolu

Laxman Dhulipala

Goran Zu

Requirements

Grading

HW	30%
Midterm	30%
Final	40%

Discussion Piazza

Web page ~glmiller

Grades A+ to B-

Audit $\equiv \geq$ B-

Course Goals

1) Understand many known:
   a) Algorithms
   b) Design Techniques
2) Analyze algorithm efficiency
3) Algorithm correctness
4) Communicate about code
5) Know key words
6) Design your own alg.

Machine Models

RAM ≡ Random Access Machine

not considered

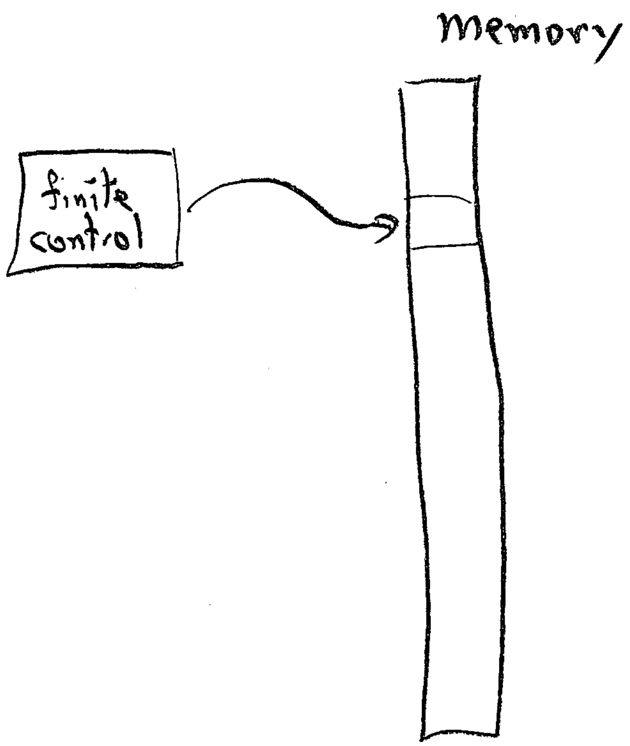
Caching models
memory hierarchy
pipelining

Parallel Models

PRAM

Circuits

RAM



v

unit time ops

memory read/write

$+, -, \times, \div$

Asymptotic Complexity

Def:

$f(n) \in O(g(n))$ if

$$\exists c, n_0 \geq 0 \; \forall n \geq n_0 \; f(n) \leq c\, g(n)$$

$$O(g(n)) = \{ f(n) \mid f(n) \in O(g) \}$$

$f \in o(g(n))$ if

$$\forall c > 0 \; \exists n_0 \geq 0 \; \forall n \geq n_0 \; f(n) \leq c\, g(n)$$

---

1) $f \in \Omega(g)$ if $g \in O(f)$

2) $f \in \Omega(g)$ if

$$\exists c > 0 \; \forall n_0 \geq 0 \; \exists n_1 \geq n_0 \; f(n_1) \geq c\, g(n_1)$$

(infinitely often)

Claim $2n^2+n+1 \in O(n^2)$

try setting $c=3$

solve $2n^2+n+1 = 3n^2$

Need $n^2-n-1 \geq 0$ OK for $n_0=2$

$\forall n \geq 2 \ (2n^2+n+1) \leq 3n^2$

L'Hopital's Rule

$\lim_{n\to\infty} \frac{2n^2+n+1}{n^2} = 2 \quad \therefore \ c = 2+\varepsilon$ works

---

Claim $3n \in o(n^2)$

I need n_0 as fcn of c

solve $3n \doteq cn^2$

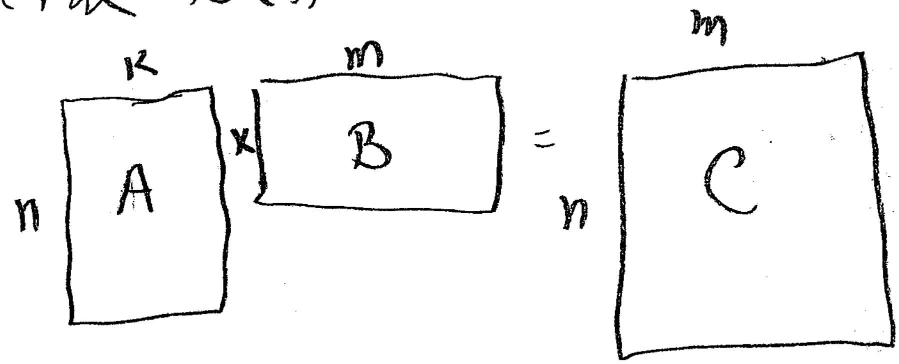
$3 \doteq cn$

$n \doteq 3/c$

set $n_0 = 3/c$

Matrix Multiplication

Several Def, Real



$$C_{ij} = \sum_{t=1}^{K} A_{it} B_{tj} \equiv A_{i*} \cdot B_{*j}$$

Facts 1) $(AB)C = A(BC)$

2) $A(B+C) = AB + AC$

3) $A \cdot B \neq B \cdot A$

4) $\lambda A \equiv \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} A$

5) $AB = C \Rightarrow C = \sum A_{*j} B_{j*}$ (outer products)

Matrix Multiplication

Naive A, B are $n \times n$ matrices over Reals

Def $A \cdot B = C$ if $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

n^3 multiplications $\quad (n-1)n^2$ additions

$O(n^3)$ operations

Recursive Alg $\quad M(A,B) \quad n = 2^k$

1) if A is 1×1 then return $a_{11} \cdot b_{11}$

2) write $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$

A_{ij} are $n/2 \times n/2 \qquad B_{ij}$ are $n/2 \times n/2$

3) $C_{ij} = M(A_{i1}, B_{1j}) + M(A_{i2}, B_{2j})$

4) return $\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$

<u>correctness</u> induction on n

$n=1$ done

assume $M(A,B) = A \cdot B$ $n < n_0$

we know $C_{ij} = A_{i1} \circ B_{1j} + A_{i2} \circ B_{2j}$

$\therefore$ $C_{ij} = M(A_{i1}, B_{1j}) + M(A_{i2}, B_{2j})$

Timing: Let $T(n)$ = number of ops for $n \times n$

$$T(n) \le 8T(n/2) + cn^2 \quad \& \quad T(1) = 1$$

Claim: $T(n) = O(n^3)$

Consider recurrence

$$T(n) = 7T(n/2) + cn^2 \quad \& \quad T(1) = 1$$

Claim: $T(n) = O(n^{\log_2 7})$

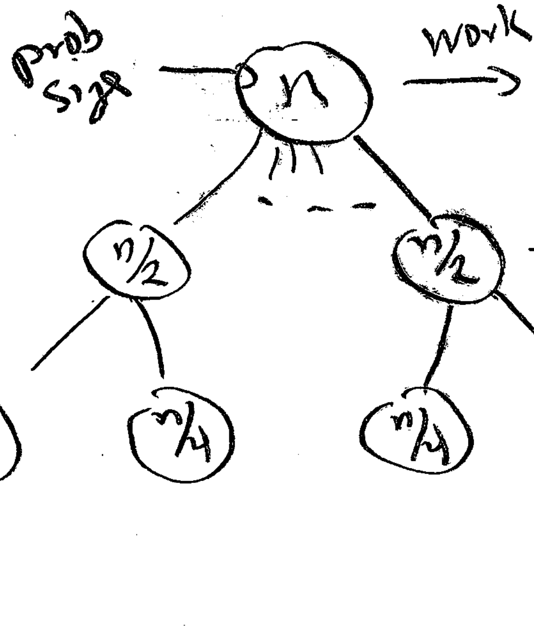
Solving Recurrences

Methods 1) Use formula

2) Induction on n

3) consider tree of recursive calls

3)



cn^2

$8c(n/2)^2 = 2cn^2$

$(2^3)^2 c\left(\frac{n}{2^2}\right)^2 = 2^2 cn^2$

(1)

(1) $2^{\log n} cn^2$

$O(n^3)$

For 7 calls

$$cn^2$$

$$7c(n/2)^2 = \frac{7}{4}cn^2$$

$$7^2c(n/4)^2 = \left(\frac{7}{4}\right)^2 cn^2$$

$$\vdots$$

$$\left(\frac{7}{4}\right)^{\log n} cn^2 = \frac{n^{\log 7}}{\cancel{n^{\log 4}}} c\cancel{n^2}$$

$$= cn^{\log 7}$$

Total $O(n^{\log 7})$

Strassen's Matrix Mult.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} S_1 + S_2 - S_4 + S_6 & S_4 - S_5 \\ S_6 + S_7 & S_2 - S_3 + S_5 - S_7 \end{bmatrix}$$

$$S_1 = (B - D)(G + H)$$

$$S_2 = (A + D)(E + H)$$

$$S_3 = (A - C)(E + F)$$

$$S_4 = (A + B) \cdot H$$

$$S_5 = A \cdot (F - H)$$

$$S_6 = D \cdot (G - E)$$

$$S_7 = (C + D) \cdot E$$

Correctness

eg $C \cdot E + D \cdot G = S_6 + S_7$

$$S_6 + S_7 = D \cdot (G - E) + (C + D) \cdot E$$
$$= D \cdot G - DE + C \cdot E + D \cdot E$$
$$= D \cdot G + C \cdot E$$

---

c_i can be found in a constant number of bit vector steps, and line 6 requires $O_{BV}(n)$. Therefore the loop of lines 1–6 is $O_{BV}(n^2/\log n)$ and line 7 is of the same complexity. □

EXERCISES

6.1 Show that the integers modulo n form a ring. That is, $\mathbf{Z}_n$ is the ring $(\{0, 1, \ldots, n-1\}, +, \cdot, 0, 1)$, where $a + b$ and $a \cdot b$ are ordinary addition and multiplication modulo n.

6.2 Show that M_n, the set of $n \times n$ matrices with elements chosen from some ring R, itself forms a ring.

6.3 Give an example to show that the product of matrices is not commutative, even if the elements are chosen from a ring in which multiplication is commutative.

6.4 Use Strassen's algorithm to compute the product

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

6.5 Another version of Strassen's algorithm uses the following identities to help compute the product of two 2×2 matrices.

$$\begin{array}{lll}
s_1 = a_{21} + a_{22} & m_1 = s_2 s_6 & t_1 = m_1 + m_2 \\
s_2 = s_1 - a_{11} & m_2 = a_{11} b_{11} & t_2 = t_1 + m_4 \\
s_3 = a_{11} - a_{21} & m_3 = a_{12} b_{21} & \\
s_4 = a_{12} - s_2 & m_4 = s_3 s_7 & \\
s_5 = b_{12} - b_{11} & m_5 = s_1 s_5 & \\
s_6 = b_{22} - s_5 & m_6 = s_4 b_{22} & \\
s_7 = b_{22} - b_{12} & m_7 = a_{22} s_8 & \\
s_8 = s_6 - b_{21} & &
\end{array}$$

The elements of the product matrix are:

$$\begin{aligned}
c_{11} &= m_2 + m_3, \\
c_{12} &= t_1 + m_5 + m_6, \\
c_{21} &= t_2 - m_7, \\
c_{22} &= t_2 + m_5.
\end{aligned}$$

Show that these elements compute Eq. (6.1). Note that only 7 multiplications and 15 additions have been used.

---

† We can get around the detail that NUM($\mathbf{a}_i$) is the integer representing the reverse of $\mathbf{a}_i$ by taking the "jth row" of B_i to be the jth row from the bottom instead of the top as we have previously done.

What is a Space Efficient Strassen?

1) Add in place
2) Malloc $3n^2$ space per call.
3) Do final additions in output space of parent.

$W(n)$ be space used

$$W(n) = 3n^2 + W(n/2)$$
$$= 3n^2 + 3(n/2)^2 + 3(n/4)^2 + \cdots$$
$$= 3n^2(1 + 1/4 + 1/16 + \cdots)$$

---

note $1+\alpha+\alpha^2+\cdots = \frac{1}{1-\alpha}$ $\quad \alpha<1$

$$= 3n^2\left(\frac{1}{1-1/4}\right) = 3n^2(4/3) = 4n^2$$