Probability 101

15-750
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2/10/16

Random Variable: Samples $\rightarrow \mathbb{R}$

Prob Density Fcn

eg X_β exponential random variable

$$\text{Prob}[X_\beta = u] = \begin{cases} \beta e^{-\beta x} & u \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

Cumulative Dist

$$F_{X_\beta}(y) = \text{Prob}[X_\beta \leq y]$$

Fund thm of cal

$$F_\beta(y) = \int_0^y \beta e^{-\beta x}\,dx \equiv -e^{-\beta x}\Big]_0^y = 1 - e^{-\beta y}$$

Def Expected Value

$$Ex[X] = \int_{-\infty}^{\infty} X \Pr[X = y]\,dy$$

$$\mathrm{Ex}[X_\beta] = \int_0^\infty y\beta e^{-\beta y}\,dy = 1/\beta$$

Intergration by parts

$$\Pr[X > m+n \mid X \geq m] = \frac{e^{-\beta(n+m)}}{e^{-\beta m}} = e^{-\beta n}$$

Memoryless Property

$\mathrm{Select}_k(x_1, \dots, x_n)$ = kth smallest from $\{x_1, \dots, x_n\}$

Order Statistics

Random Variables $X_1 \dots X_n$

$X_{(k)} \equiv \mathrm{Select}_k(X_1, \dots, X_n)$

Note $X_{(1)} \leq \dots \leq X_{(n)}$

$$P_r[X_{(1)} = u] \equiv f_1(u) \qquad f(u) = P_r[X_i = u] \text{ for any } i$$

$$P_r[0 \le X_i \le u] = F(u)$$

$$f_1(u) = n(1 - F(u))^{n-1} f(u) = n e^{-(n-1)\beta u} \beta e^{-\beta u}$$

$$= n\beta e^{-n\beta u} \equiv \text{Exp}(n\beta)$$

$$\text{Ex}[f_1] = \frac{1}{n\beta}$$

$$S_i \equiv X_{(i+1)} - X_{(i)} \qquad i \ge 0$$

by memoryless prop $S_i \equiv \text{Exp}((n-i)\beta)$

$$\text{Ex}[S_i] = \frac{1}{(n-i)\beta}$$

$$\text{Ex}[X_{(n)}] = \sum_{i=0}^{n-1} \text{Ex}[S_i] = \frac{1}{\beta}\left[1 + \frac{1}{2} + \cdots \frac{1}{n-1}\right] \approx \frac{\lg n}{\beta}$$

Concentration for $X_{(n)}$

$$\Pr\left\{X_i \geq \frac{c \ln n}{\beta}\right\} = e^{-\frac{c \ln n}{\beta}\beta} = n^{-c}$$

$$\Pr\left[X_{(n)} \geq \frac{c \ln n}{\beta}\right] \leq n \cdot n^{-c} = \frac{1}{n^{c-1}}$$

$$\Pr\left[X_{(n)} \geq \frac{2 \ln n}{\beta}\right] \leq \frac{1}{n}$$

Generating Dist

$f : \mathbb{R} \to \mathbb{R}^+$ PDF $\int_{-\infty}^{\infty} f(x)\,dx = 1$ X_f random variable

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Def $f, g \in$ PDF's we say $f \leq g$ if

$\exists$ det process D s.t. $X_f = D(X_g)$

$\mathcal{U}$ is uniform $[0,1]$ ie $u(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{o.w.} \end{cases}$

$\mathcal{U}_2$ is uniform $[0,2]$

Claim $U_2 \leq U$ $X_{U_2} = 2X_U$

Generate Exponential Dist from uniform.

Pdf $\equiv$ $f(x) = \beta e^{-\beta x}$ $\quad 0<\beta$ & $x \geq 0$

$$F(x) = \int_0^x f(x)\,dx = 1 - e^{-\beta x}$$

$F: [0, \infty] \to [0, 1]$ 1-1 & onto

Thus $U \leq f$

Random Var $Y = F(X) = 1 - e^{-\beta X}$

$$e^{-\beta Y} = 1 - Y$$

$$-\beta X = \ln(1-Y)$$

$$X = -\frac{1}{\beta}\ln(1-Y)$$

$$X = \frac{-\ln(\mathcal{U})}{\beta}$$

Note Y uniform $[0,1]$ then
$1-Y$ " "

Exp $\leq \mathcal{U}$

Normal Dist

PDF $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$

$\sigma = \frac{1}{\sqrt{2}}$ $f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ Gauss's Normal

(Unit Normal $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$)

What is the cumulative!

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dx$$

Thm $F(x)$ not an elementary fcn!

See Adamchik's class (Risch Intergration)

note $f = x e^{-x^2/2}$ OK $\frac{d\,{-e^{-x^2/2}}}{dx} = x e^{-x^2/2}$

Consider 2D normal $f(x,y) = \frac{1}{2\pi} e^{-x^2/2} \cdot e^{-y^2/2}$

$$f(x,y) = \frac{1}{2\pi} e^{\frac{-(x^2+y^2)}{2}}$$

In polar $f(r,\theta) = \frac{1}{2\pi} e^{-r^2/2}$ (symmetric)

Cumulative is prob in a disk of radius r.

$$D(R) = \int_0^R \frac{2\pi}{2\pi} r e^{-r^2/2} dr = \int_0^R r e^{-r^2/2} dr$$

$$-e^{-r^2/2}\Big]_0^R = 1 - e^{-R^2/2}$$

$Y = 1 - e^{-R^2/2}$ $e^{-R^2/2} = 1-Y$ $-\frac{R^2}{2} = \ln(1-Y)$ $R = \sqrt{-2\ln(1-Y)}$	Alg u, v uniform $[0,1]$ in polar return (r,θ) where $r = \sqrt{-2\ln u}$ $\theta = 2\pi v$

or return $x = r\cos\theta$

$y = r\sin\theta$

Another Method (Box-Muller)

1) u, v uniform $[0,1]$

   $U = 2u-1$ & $V = 2v-1$ uniform $[-1,+1]$

2) Set $W = U^2 + V^2$

   if $W > 1$ restart

3) $A = \sqrt{(-2\ln W)/W}$

4) return $T_1 = UA$ & $T_2 = VA$.

BM Pf:

$V_1 = R\cos\theta$
$V_2 = R\sin\theta$
$S = R^2$

$$X_1 = V_1\sqrt{\frac{-2\ln S}{S}} = (R\cos\theta)\sqrt{\frac{-2\ln S}{R^2}} = \sqrt{-2\ln S}\cos\theta$$

$$X_2 = \sqrt{-2\ln S}\sin\theta$$

In polar $X_1 = R'\cos\theta'$ $X_2 = R'\sin\theta'$

ie $R' = \sqrt{-2\ln S}$ & $\theta' = \theta$ (indep)

$$\Pr[R' \le r] = \Pr[-2\ln S \le r^2]$$
$$= \Pr[S \ge e^{-r^2/2}] = 1 - e^{-r^2/2}$$

S = uniform $[0,1]$?

PDF R is $f_R(r) = 2r$ $\quad F_R(x) = \int_0^x 2r\,dr = r^2\Big]_0^x = x^2$

CDF for R^2 $\quad F_{R^2}(x) = \sqrt{x^2} = x$

PDF for R^2 $\quad f_{R^2} = 1$

PDF R' $\quad f_{R'}(r) = r e^{-r^2/2}$