LP-Duality

An amazing fact!

LPs have duals.

We get Weak & Strong duality Thms.

Start with Standard form:

Primal (*)	Dual (**)	or
Max $c^T x$	Min $b^T y$	Min $y^T b$
Subject $Ax \le b$	Subject $A^T y \ge c$	Subject $y^T A \ge c^T$
$x \ge 0$	$y \ge 0$	$y^T \ge 0$

Weak Duality Thm

If x feasible for (*) & y feasible for (**)

then $c^T x \le b^T y$

pf Consider $y^T A x$

Given $Ax \le b$ & $y^T \ge 0$ $\Rightarrow$ $y^T A x \le y^T b$

Given $y^T A \ge c^T$ & $x \ge 0$ $\Rightarrow$ $c^T x \le y^T A x$

Thus $c^T x \le y^T A x \le y^T b$

Another form

Primal	Dual
Max $c^T x$	Min $y^T b$
Subject $Ax \leq b$	Subject $y^T A = c^T$
	$y^T \geq 0$

(Weak Duality) x feasible & y feasible then $c^T x \leq y^T b$

pf $\quad c^T x = \underbrace{y^T A}\, x = y^T \underbrace{Ax} \leq y^T b$

(Strong Duality) $\exists$ feasibe x & y st

$$c^T x = y^T b$$

thus x & y are optimal!

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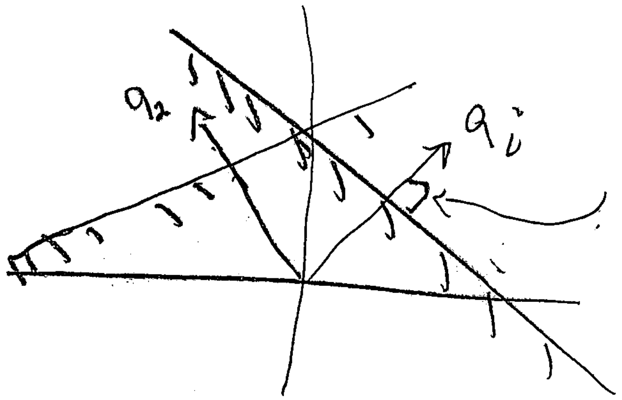
pf Geometric Pf:

Suppose $A^{m \times n} = \begin{pmatrix} -a_1^T- \\ \vdots \\ -a_m^T- \end{pmatrix}$ $a_i^T \equiv$ ith row of A.

Consider case $n=2$
(we do a 2D pf)

Geometric interpretation of each row.

Say $a_i^T x \le b_i$



a_i normal to half space

Thus $\{x \mid Ax \le b\}$ convex polytope
with rows of A as normals.

What is the dual?

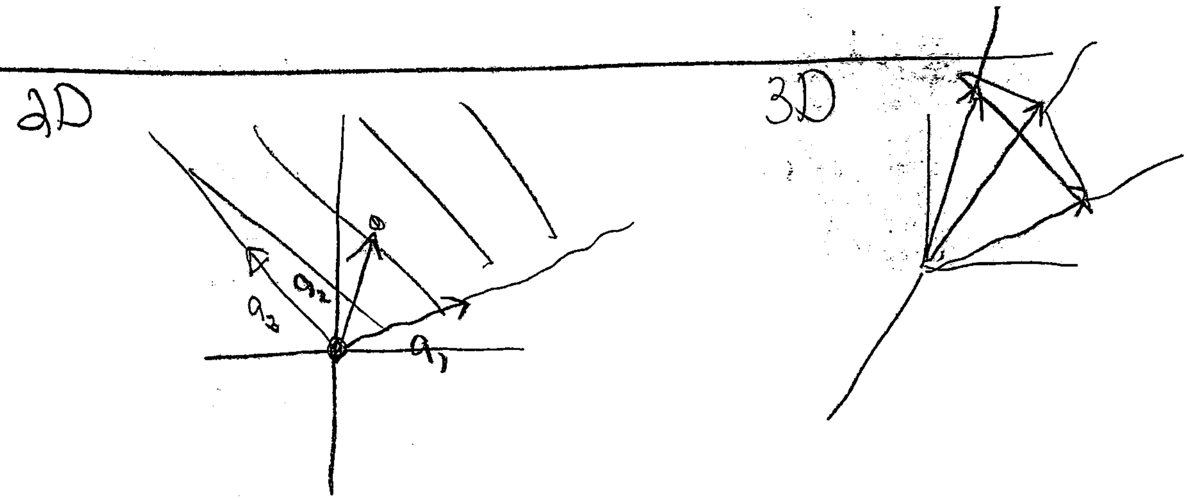
$$\{ A^T y \mid y \geq 0 \}$$

$$= \{ y_1 a_1 + \cdots y_m a_m \mid y_1, \ldots, y_m \geq 0 \}$$

1) non-negative combinations of $a_1, \ldots, a_m$.

2) a cone formed from $a_1, \ldots, a_m$!

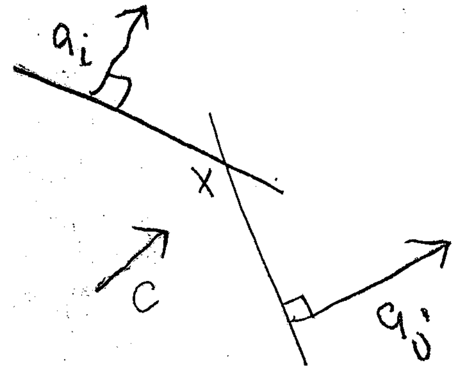
eg



Thus the constraints $A^T y = c$ & $y \geq 0$

Says: Write c as non-negative combination of $\begin{matrix} | & & | \\ a_1 & \cdots & a_m \\ | & & | \end{matrix}$

Suppose that x is optimal for primal



$a_i^T x = b_i$ & $a_j^T x = b_j$

thus $c \in \text{Cone}(a_i, a_j)$

$\therefore \exists y_i$ & y_j s.t. $y_i a_i + y_j a_j = c$

$y_i, y_j \geq 0$

Consider $y^T = (0, \ldots, 0, y_i, 0 \cdots 0, y_j, 0, \ldots, 0)$

Claim $y^T A x = y^T b = y_i b_i + y_j b_j$

pf

By optimality of x $\quad Ax = \begin{pmatrix} \vdots \\ b_i \\ \vdots \\ b_j \\ \vdots \end{pmatrix}$

$$y^T A x = (0 \cdots 0, y_i, 0 \cdots 0, y_j, 0 \cdots 0) \begin{pmatrix} \vdots \\ b_i \\ \vdots \\ b_j \\ \vdots \end{pmatrix} = y_i b_i + y_j b_j$$

$$c^T x = y^T A x = y^T b. \quad (SD)$$

This pf works in any dim!

Back to s-t Max-Flow
and s-t Min-Cut.

Recall Original LP for MaxFlow

$$\max \sum_{(s,v)\in E} f(s,v)$$

subject $\sum_{(u,v)\in E} f(u,v) = \sum_{(v,w)\in E} f(v,w) \quad \forall v \in V \setminus s,t$

$$f(u,v) \leq c(u,v)$$
$$f(u,v) \geq 0$$

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note #varibles = # edges = m

#constraints = # vertices − 2 + m

$n-2+m$

thus dual has too many variables to be simple min cut!

Not in standard form!

We consider LP with fewer constraints

But exponential # vars!

But in Standard form.

Let $P \equiv$ set of all simple paths from s to t

eg 

for each $p \in P$ introduce var X_p.

Path Form

$$\max \sum_{p \in P} X_p$$

$$\text{subject: } \sum_{p \in P \wedge (u,v) \in p} X_p \le c(u,v) \quad \forall (u,v) \in E$$

$$X_p \ge 0 \quad \forall p \in P$$

We have m constraints!

Claim 1 f feasible for original then
$\exists$ a feasible path-form with same flow.

pf Write f as a flow of paths (pf?)

note We need only m path-flows for
a given flow. (Why?)

Claim 2 If $\{x_p\}_{p \in P}$ is feasible path-flow
then $\exists$ feasible flow with same flow.

pf Set $f(u,v) = \sum_{p \in P,\, (u,v) \in p} x_p$

Thus $0 \leq f(u,v) \leq c(u,v)$

$\forall v \in V \setminus s,t \quad \sum_{(u,v) \in E} f(u,v) = \sum_{p \in P\ v \in p} x_p = \sum_{(v,w) \in E} f(v,w)$

<u>Note</u> Path-form LP is in standard form thus dual

$$\min \sum_{(u,v)\in E} c(u,v)\, y_{uv}$$

$$\text{subject} \sum_{(u,v)\in p} y_{uv} \geq 1 \quad \forall p \in P.$$

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path-form

$$\begin{array}{c|c} & P_1 \; - - \; P_k \\ \begin{array}{c} e_1 \\ | \\ \\ e_m \end{array} & \begin{pmatrix} -0\,1\,1\,1\,0 - - \\ A \\ \\ \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ , \\ , \\ x_k \end{pmatrix} \leq \begin{pmatrix} c_1 \\ , \\ . \\ c_m \end{pmatrix}$$

$$A_{ij} = \begin{cases} 1 & \text{if } e_i \in P_j \\ 0 & \text{o.w.} \end{cases} \qquad \max \sum x_i$$

$$(y_1 - - - y_m)\, A \geq (1, - - , 1)$$

We think of $y_{uv} \equiv$ length of (u,v).

Condition $\sum_{uv \in P} y_{uv} \geq 1 \Rightarrow dist(s,t) \geq 1$

We may assume $dist_y(s,t) = 1$

Fix a Min Cut of G

Claim Opt $\equiv \hat{y}_{uv} = \begin{cases} 1 \text{ if } (uv) \in \text{Min Cut} \\ 0 \quad \text{o.w.} \end{cases}$

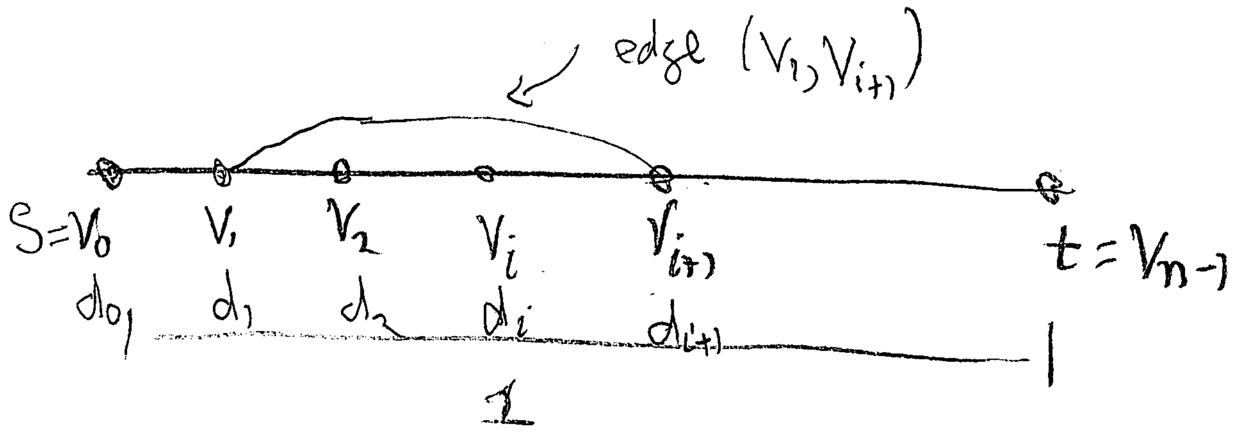
pf

Note $\hat{y}$ is feasible & $\sum c_{uv} \hat{y}_{uv} = |\text{Min Cut}|$

to Show $\forall$ feasible y $\quad c^T y \geq |\text{Min Cut}|$

pf Let y be feasible st $dist_y(s,t) = 1$

Sort vertices by dist to s



Def $\quad Cut_i = \sum_{k \le i < l} C(v_k, v_l)$

$$C^T Y = \sum (d_{i+1} - d_i)\, Cut_i \geq \left[\sum (d_{i+1} - d_i)\right] (\text{Min Cut})$$

$$= (\text{Min Cut})$$