15-750
4/27/16

Today

Assume basic NP-Completeness

Let L be a language

L ∈ NP if ∃ short proof of membership

L ≡ all 3-colorable graphs

the prove G is 3-color simply exhibit 3-coloring.

$L \leq_p L'$ if ∃f st 1) $x \in L$ iff $f(x) \in L'$

2) f poly time.

Def: G is graph H ⊆ G is a Hamiltonian Cycle

if H is simple cycle containing V.

Approximation Algorithms

15-451
5/3/12
CLRS Chap 35

Goal: Solve NP-Hard Problems Approximately

We now consider optimization problems

e.g. 1) Coloring a graph with min # of colors
2) Finding a min size vertex cover

Let P be an optimization prob.

<u>Def</u> An alg A is poly-time <u>K-approx</u> alg if.
1) A is poly-time
2) $A(\tau) \le K \, Opt_P(\tau)$

Vertex-Cover Prob

Input: $G=(V,E)$

Output: Minimum size $C \subseteq V$ that "covers" E

Question: How do we lower bd Opt?

One Answer for VC:

<u>Def</u> $e, e' \in E$ are <u>independent</u> if $e \cap e' = \emptyset$

Let $E' \subseteq E$ be an ind set of edges.

<u>Claim</u> $Opt \geq |E'|$

<u>pf</u> Opt must contain at least one endpoint from each $e \in E'$.

Alg Approx-VC

1) Find a maximal ind set of edges E'

2) Return V' the endpoints of E'.

Note Approx-VC returns a vertex-cover.

Claim Approx-VC is a poly-time 2-approx alg.

pf 1) Clearly Approx-VC is poly-time

2) $|\text{Approx-VC}| = 2|E'| \leq 2|\text{Opt-VC}|$

$(2-\varepsilon)$-approx thought hard.

Traveling - Salesman Prob

Input: Complete graph $G=(V,E)$
Cost fcn $C: E \to R^+$
Output: Min-cost Hamiltonian cycle.

Claim TSP is NP-Hard

pf Ham-Cycle $\leq_P^T$ TSP

$$G=(V,E) \Rightarrow C(u,v) = \begin{cases} 0 & \text{if } (u,v) \in E \\ 1 & \text{o.w.} \end{cases}$$

$$G \in \text{Ham-Cycle iff } |TSP(C)| = 0$$

TSP with Triangle Inequality

Def c satisfies the triangle inequality if

$$\forall u, v, w \in V \quad c(u,w) \leq c(u,v) + c(v,w)$$

Claim TSP with tri-inequality is NP-Hard.

Pf Ham-Cycle $\leq_p^T$ TSP with TI

$G = (V,E)$ $\quad C((u,v)) = \text{dist}(u,v)$ in G.

$\text{Opt-TSP}(G,c) = n$ iff $G \in$ Ham Cycle

Question: Lower Bound Opt-TSP?

Let T^* be Opt Tour.

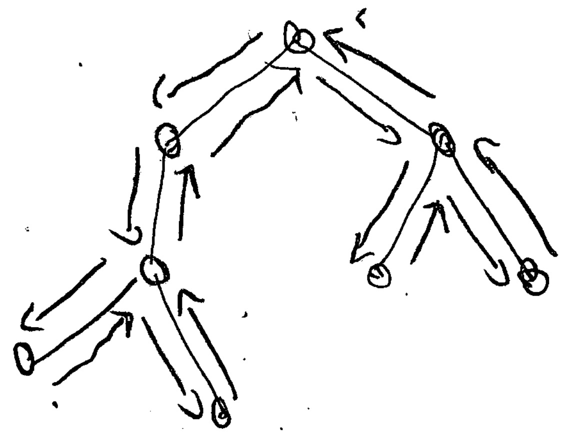
Thus T^* minus some edge is a spanning Tree.

$\therefore \ |MST| \leq |T^*|$

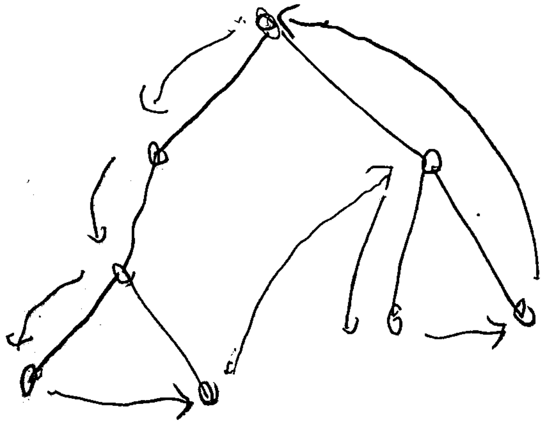
Approx-TSP Alg

1) Find a MST T of G
2) Compute an Euler Tour of T.
3) Remove multiply visited vertices from Euler Tour.

EG



Euler Tour



Ham-Cycle

Let Tour be an Euler-Tour with extra vertices shunted.

$$|\text{Tour}| \underset{\Delta}{\le} |ET| \le 2|MST| \le 2 \cdot \text{Opt-TSP}$$

<u>Thm</u> If $P \neq NP$ then $\forall \rho > 1$ $\nexists$ poly-time ρ-approx for TSP.

pf

To Show: Ham-Cycle $\leq_p^T$ ρ-approx TSP

Input: $G=(V,E)$

Output: $G'=(V,E')$ $E'=\{(u,v) \mid u \neq v\}$

$$C(u,v) = \begin{cases} 1 & \text{if } (u,v) \in E \\ \rho \cdot n + 1 & \text{o.w} \end{cases} \qquad n = |V|$$

Note $G \in$ Ham-Cycle then $|TSP| = n$

$G \notin$ Ham-Cycle then $|TSP| \geq (n-1) + \rho n + 1$

$$= (\rho+1)n$$

Thus If $G \in$ Ham-Cycle & Appox TSP returns a ρ-approx it must be a Ham-Cycle.

Center Selection Prob

Input: $p_1 \cdots p_n \in \mathbb{R}^d$ sites, int k

Output: $C = \{c_1, \cdots, c_k\} \subseteq \mathbb{R}^d$ centers

st $\min \max_{p \in P} \text{dist}(p, C) = r$

Def $\text{dist}(p, C) = \min_{c \in C} \text{dist}(p, c)$

Note $d=2$ & $k=1$ we gave a linear time alg.

In general: NP-Hard

Goal: 2-approx ie $2r$-solution.

2 algorithms

1) 2-approx given r.
2) 2-approx without r.

Simple Greedy $(P=\{p_1,\dots,p_n\}, r)$ $n>k$

Init: $C=\emptyset$, $Q=\{p_1,\dots,p_n\}$

While $Q\neq\emptyset$

1) extract p from Q (any p)

2) add p to C

3) remove from Q all q; $2r \geq dist(p,q)$

<u>Thm</u> Simple Greedy is a 2-Approx (given r)

<u>proof</u> induction on k

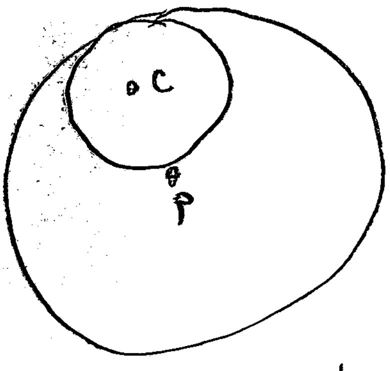
If $k=1$ done since dia is $\leq 2r$

Assume true for $k-1$.

Initially: P has k-centers of radius r solution.

after one round of while loop

P will have $(k-1)$-centers of radius r solution.



since we removed a center and points in it.

Furthest Greedy (P, k)

Init: $C = \{P_1\}$ $P = \{P_1, \dots, P_n\}$ $k = 1$

While $k \neq 0$

1) Pick $p \in \{P_1, \dots, P_n\}$ max $dist(P, C)$

2) Add p to C

3) $k = k - 1$

Thm Furthest Greedy is a 2-Approx

pf While $\max_{p \in P} dist(P, C) > 2r$ points picked could have been pick by Simple Greedy

If $\max_{p \in P} dist(p, C) \leq 2r$ done.