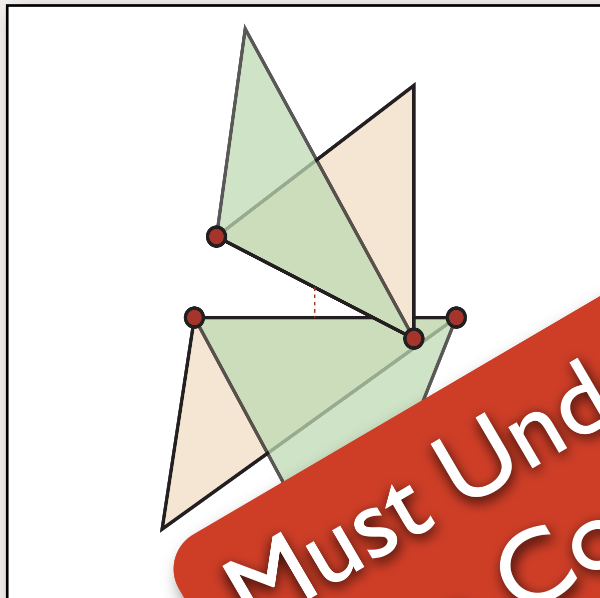
Partial Differential Equations and Fluids

Adrien Treuille

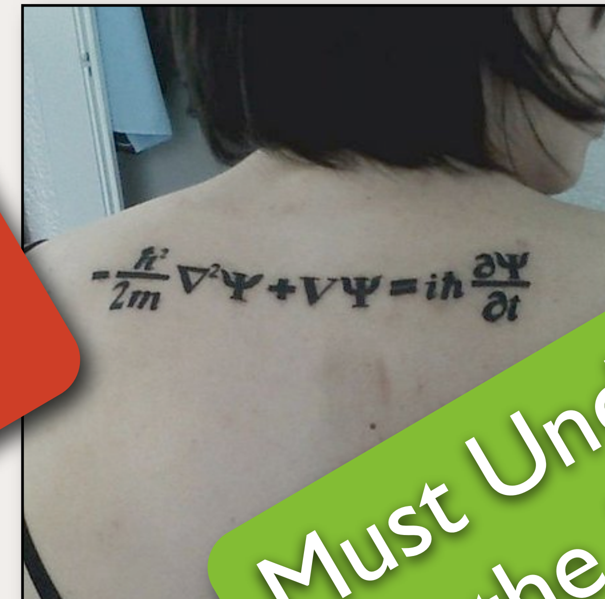
source: http://talklikeaphysicist.com/

Outline



Cloth

Must Understand the Concepts!



Must Understand the Math!

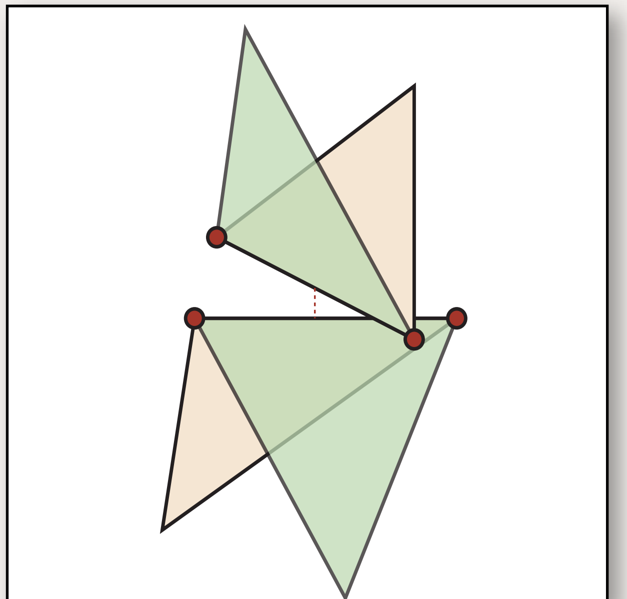
Fluid

Must Understand the Concepts!

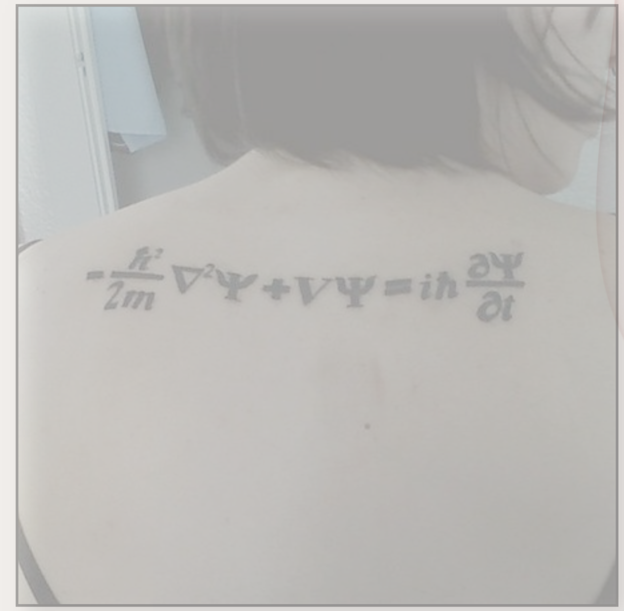
Fluid Sim

Must Understand the Concepts!

Outline

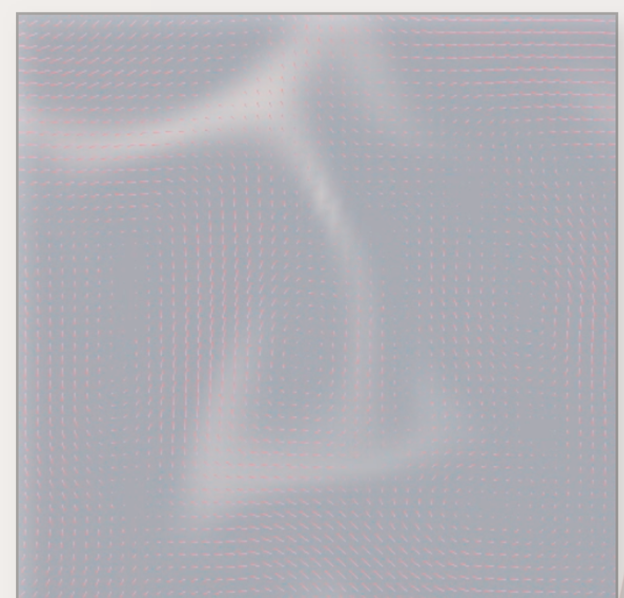


Cloth Collisions



PDEs

Fluid Physics

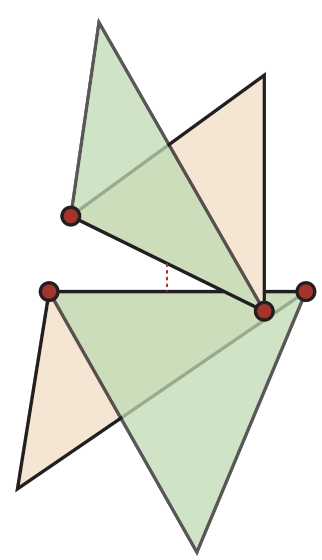


Fluid Simulation

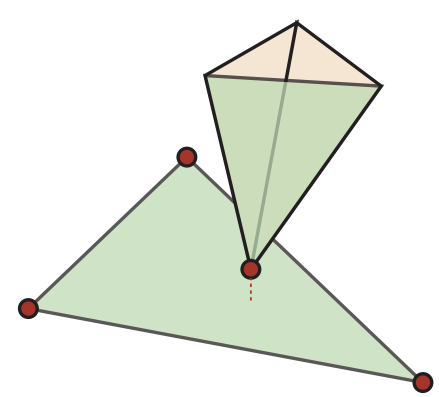
Collision Detection

Collision detection

- Generally need to do triangle-triangle collision checks:



Edge-edge collision

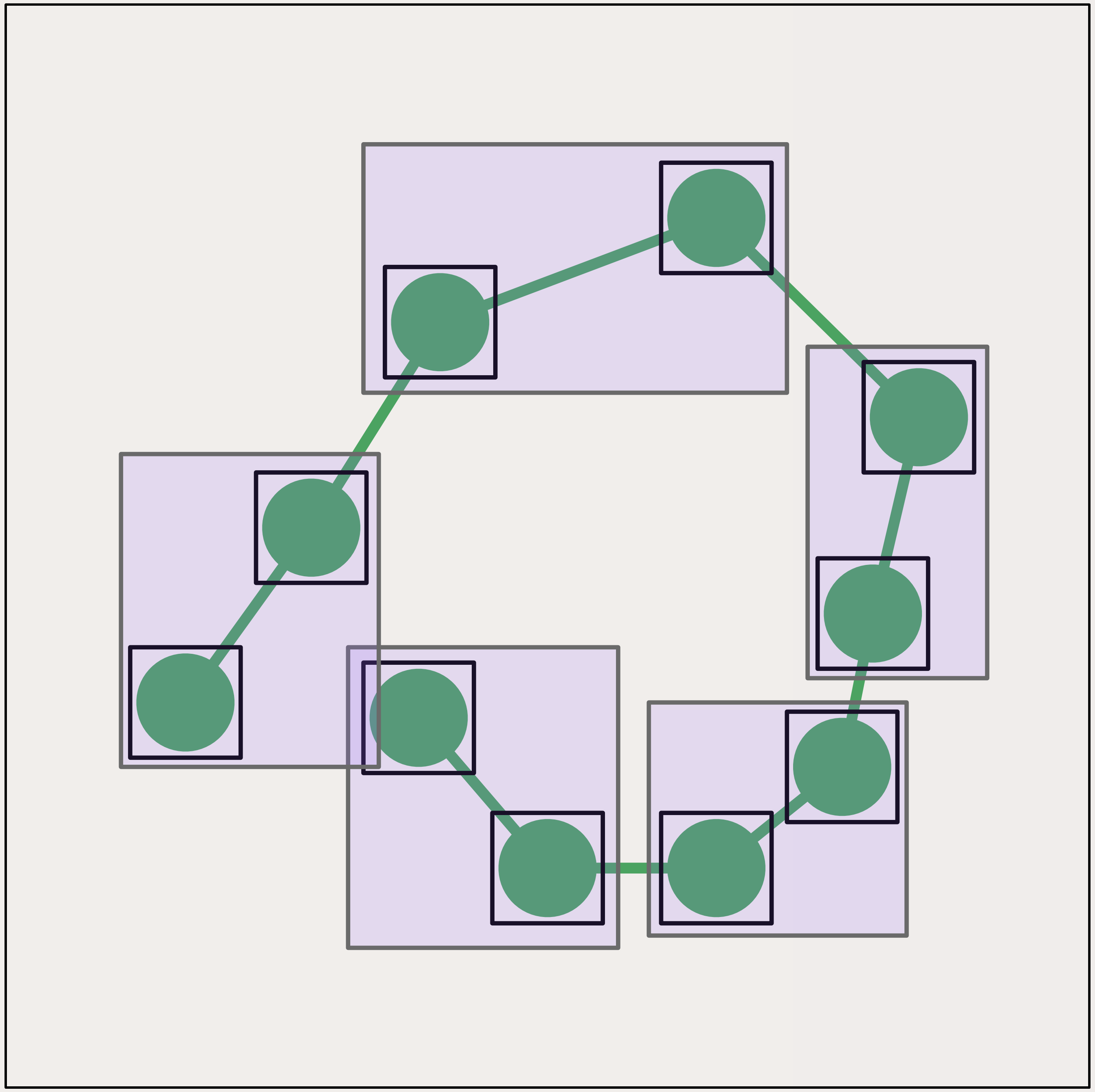


Point-face collision

Cloth Animation

Christopher Twigg
March 4, 2003

Cloth Collision Detection



Curvature Optimization

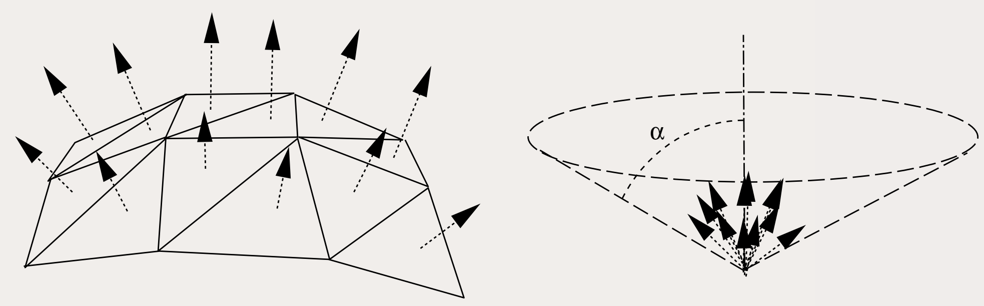


Figure 1: Cone including normals to triangles of a zone of the cloth surface.

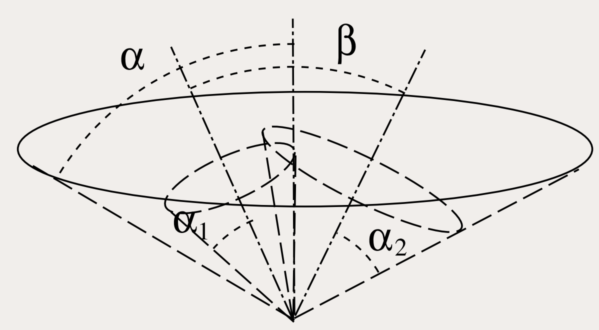


Figure 2: Cone (angle α) enclosing its two “descendant” cones in the hierarchical tree (angles α_1 and α_2).

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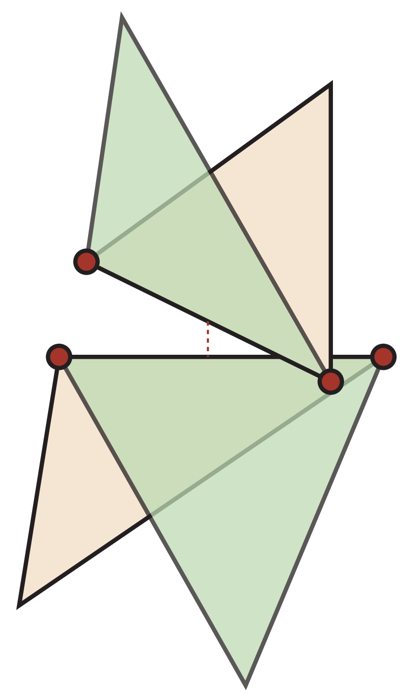
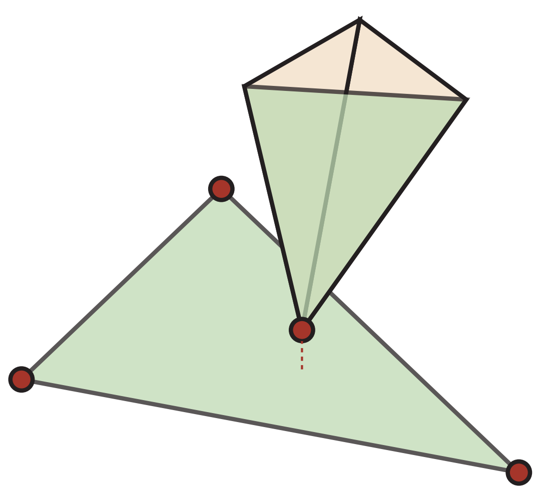
Collision detection

- We already covered this for deformable bodies
- Many of the same methods work, especially acceleration methods
- Generally need to do triangle-triangle collision checks:



Edge-edge collision



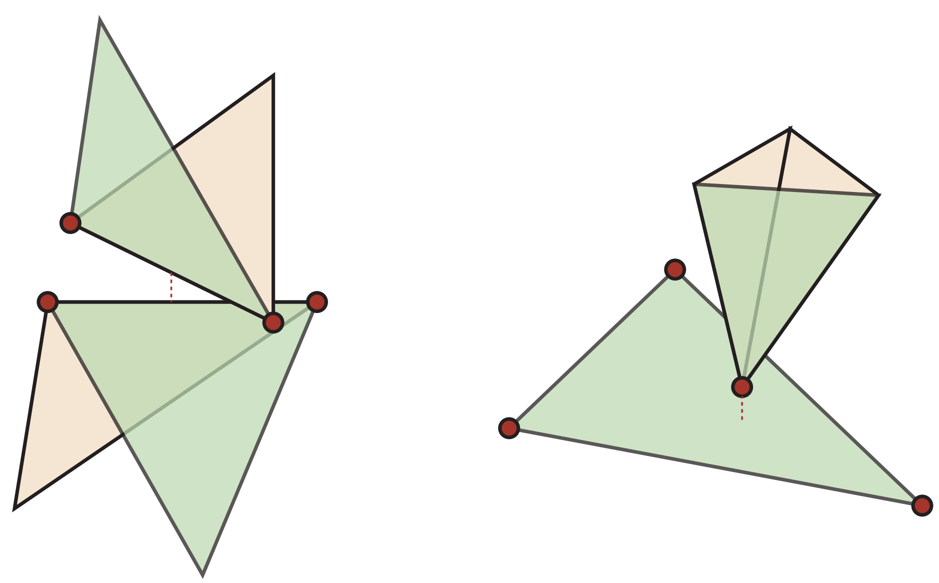
Point-face collision

Robust collision detection

If triangles are moving too fast, they may pass through each other in a single timestep.

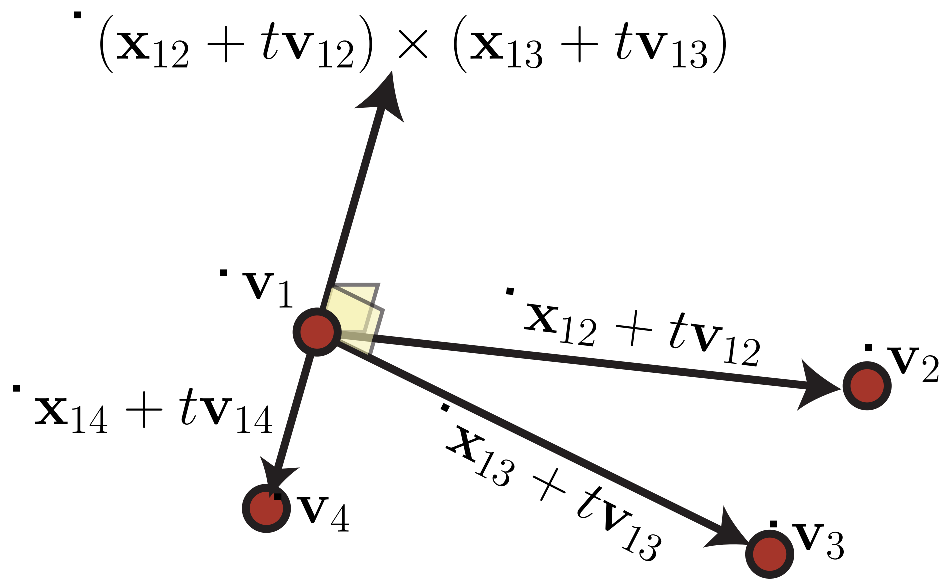
We can prevent this by checking for *any* collisions during the timestep (Provot [1997])

Note first that both point-face and edge-edge collisions occur when the appropriate 4 points are *coplanar*



Robust collision detection (2)

Detecting time of coplanarity - assume linear velocity throughout timestep:



So the problem reduces to finding roots of the cubic equation

$$\left((\mathbf{x}_{12} + t\mathbf{v}_{12}) \times (\mathbf{x}_{13} + t\mathbf{v}_{13})\right) \cdot (\mathbf{x}_{14} + t\mathbf{v}_{14})$$

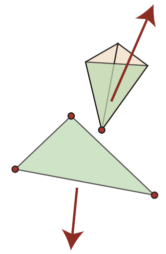
Once we have these roots, we can plug back in and test for triangle adjacency.

Collision Response

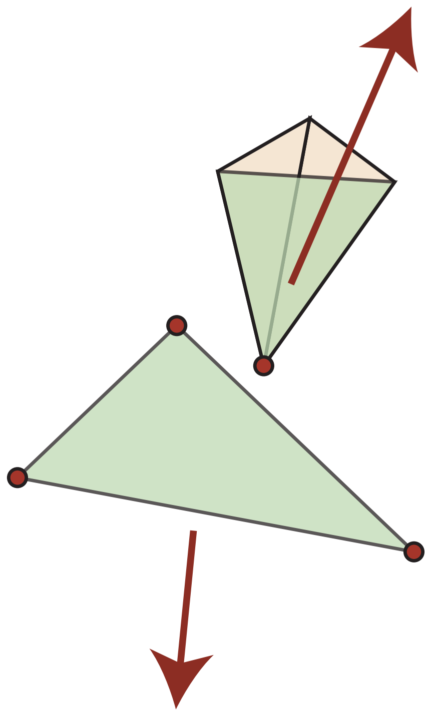
Cloth Animation

Christopher Twigg
March 4, 2003

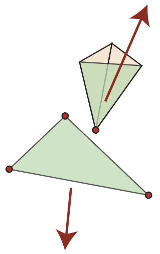
Collision response



- 4 basic options:
  - Constraint-based
  - Penalty forces
  - Impulse-based
  - Rigid body dynamics (will explain)



Collision response



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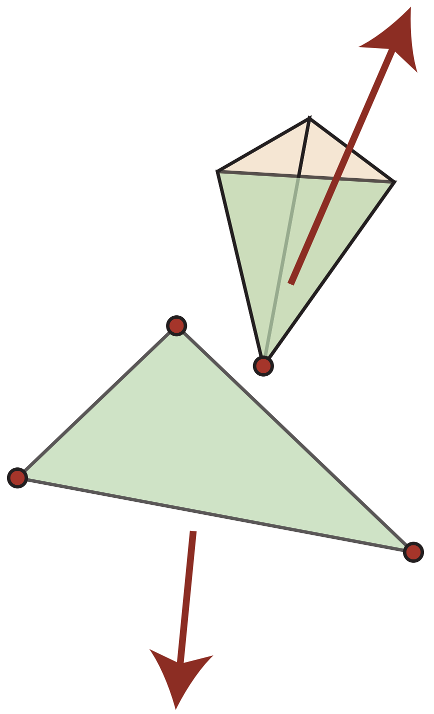
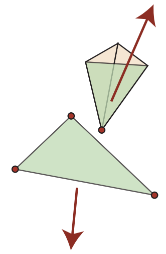


Collision response



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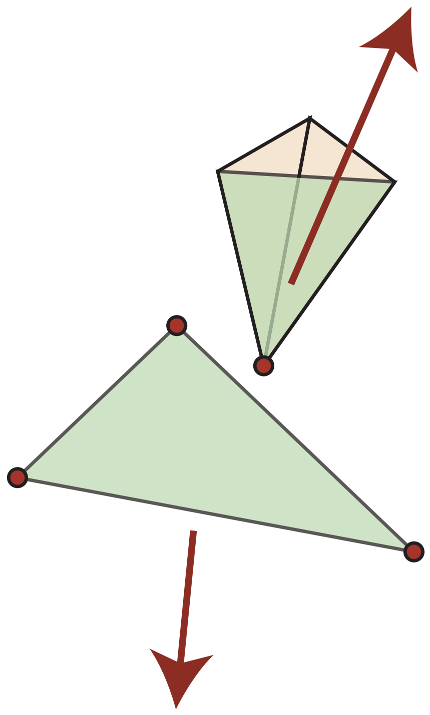
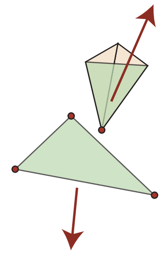
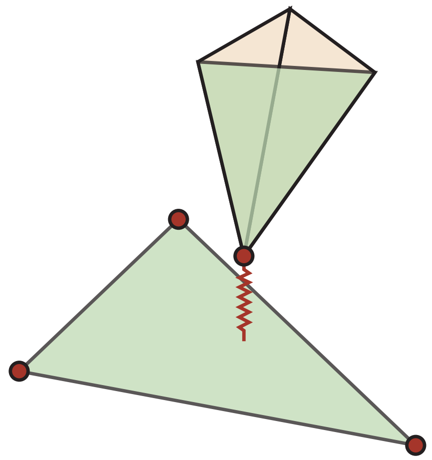


Penalty forces



- Apply a spring force that keeps particles away from each other
- Benefits:
  - Easy to fit into an existing simulator
  - Works with all kinds of collisions (use barycentric coordinates to distribute responses among vertices)

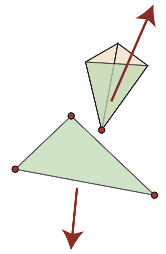


- Drawbacks:
  - Hard to tune: if force is too weak, it will sometimes fail; if force is too strong, it will cause the particles to “float” and “wiggle”

Penalty forces (2)

- In general, penalty forces are not inelastic (springs store energy)
- Can be made less elastic by limiting force when particles are moving away
- Some kind of additional damping may be needed to control deformation rate along surface

Collision response



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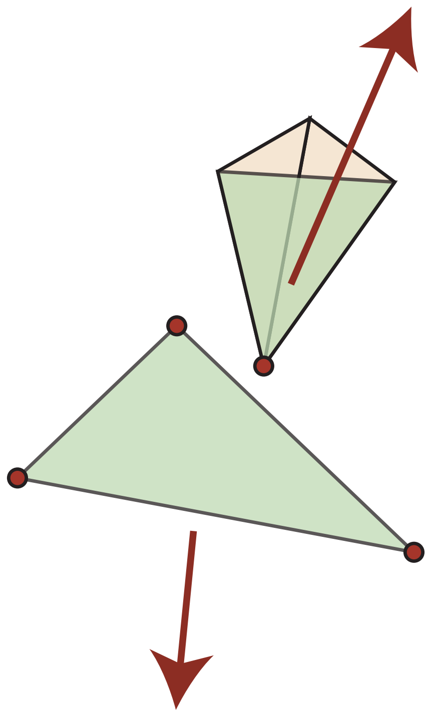

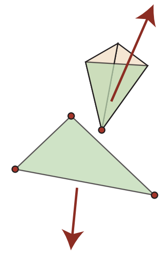

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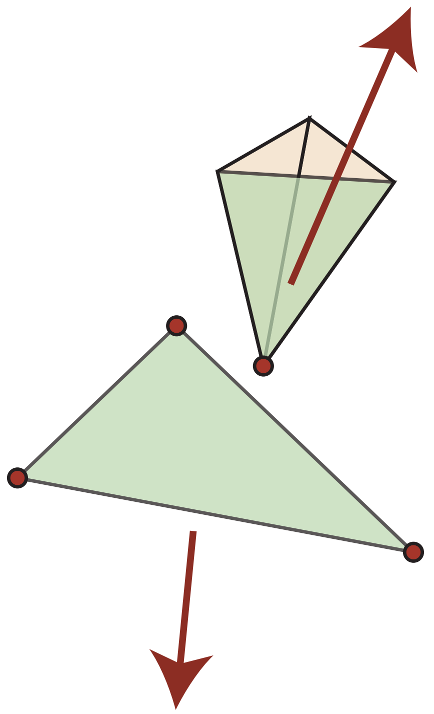


Impulses

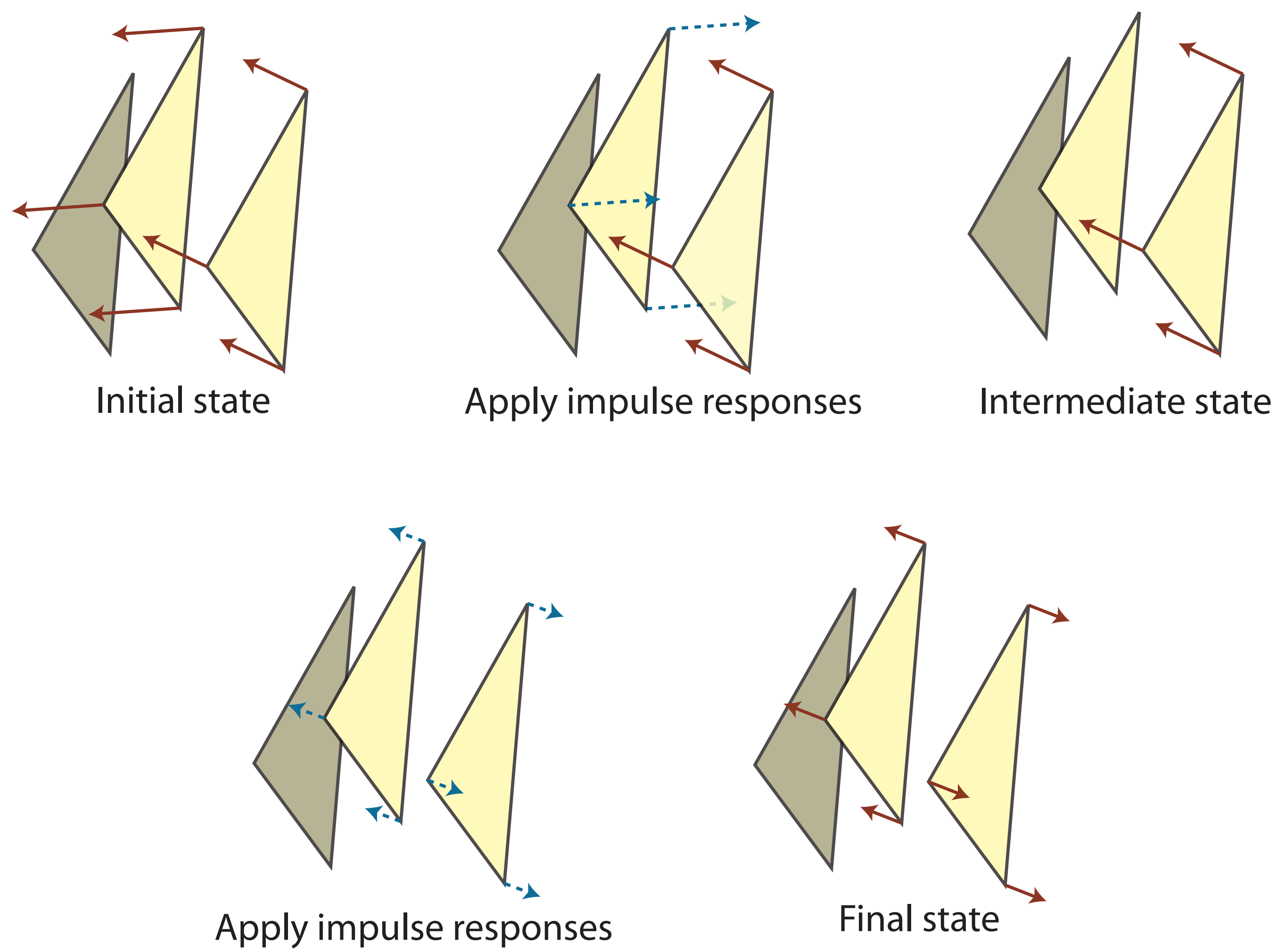
- "Instantaneous" change in momentum

$$\mathbf{J} = \int_{t_i}^{t_f} \mathbf{F}\, dt = \mathbf{p}_f - \mathbf{p}_i$$

- Generally applied outside the simulator timestep
- Benefits
  - Correctly stops all collisions (no sloppy spring forces)
- Drawbacks
  - Can have poor numerical performance
  - Handles persistent contact poorly

Impulses (2)

Iteration is generally necessary to remove all collisions.



Convergence may be slow in some cases.

Collision response

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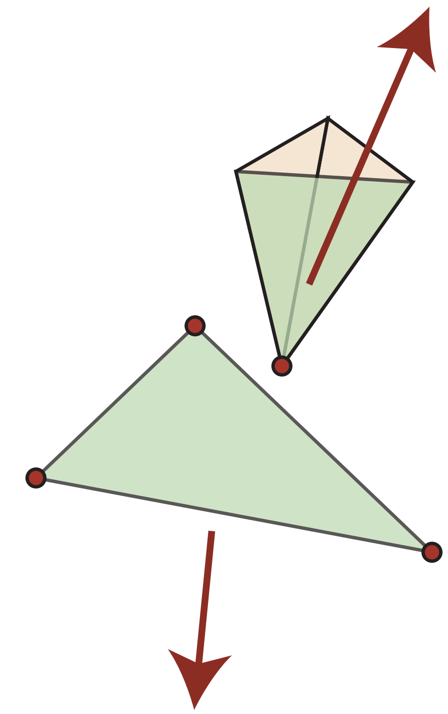
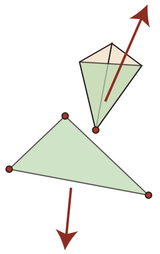


Collision response



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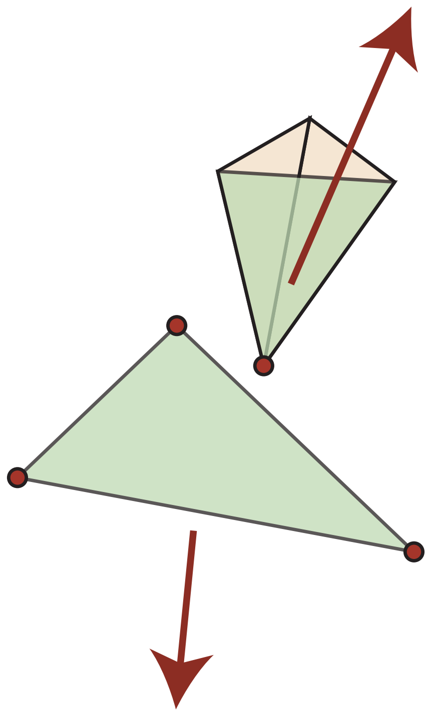

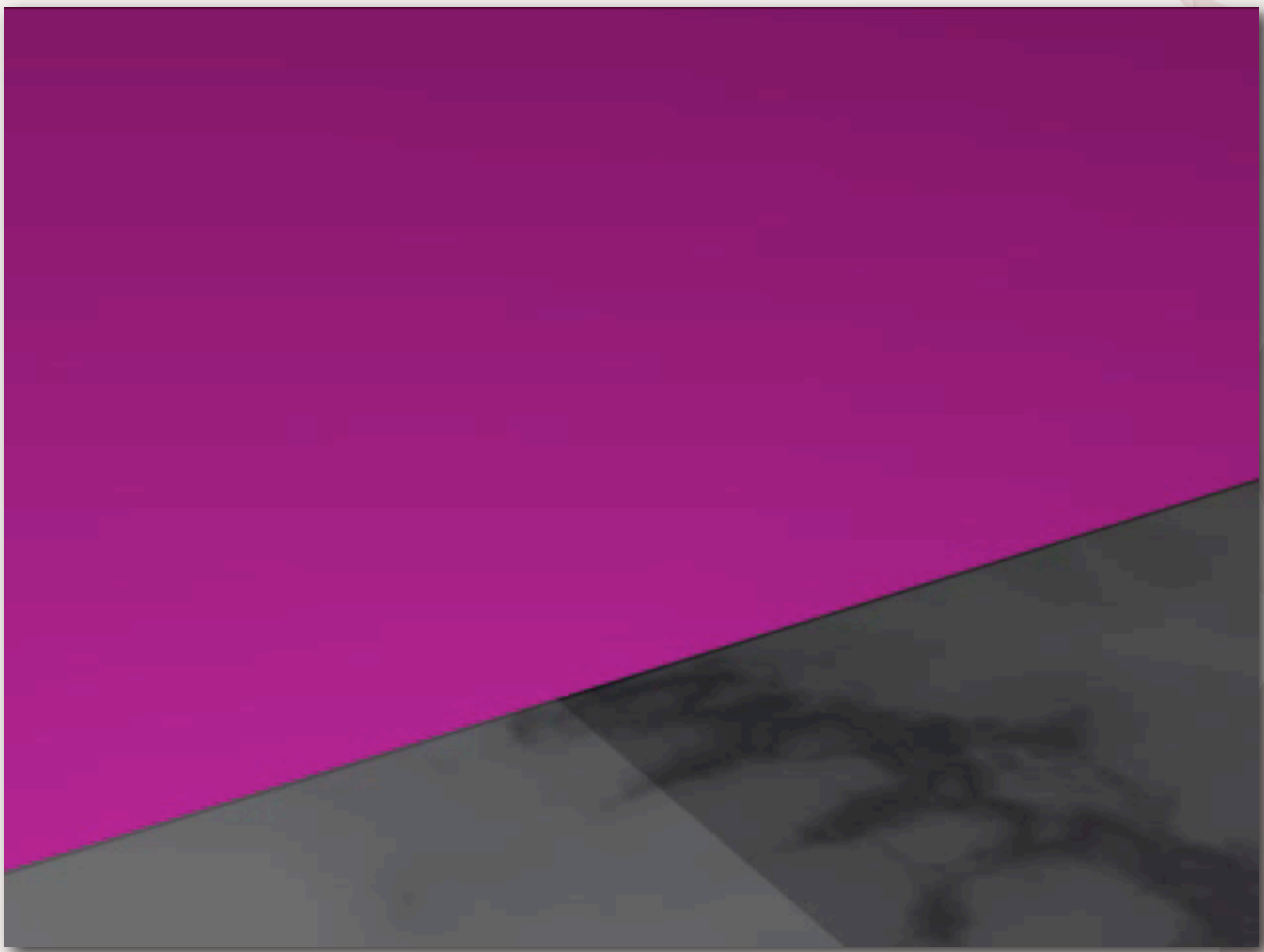

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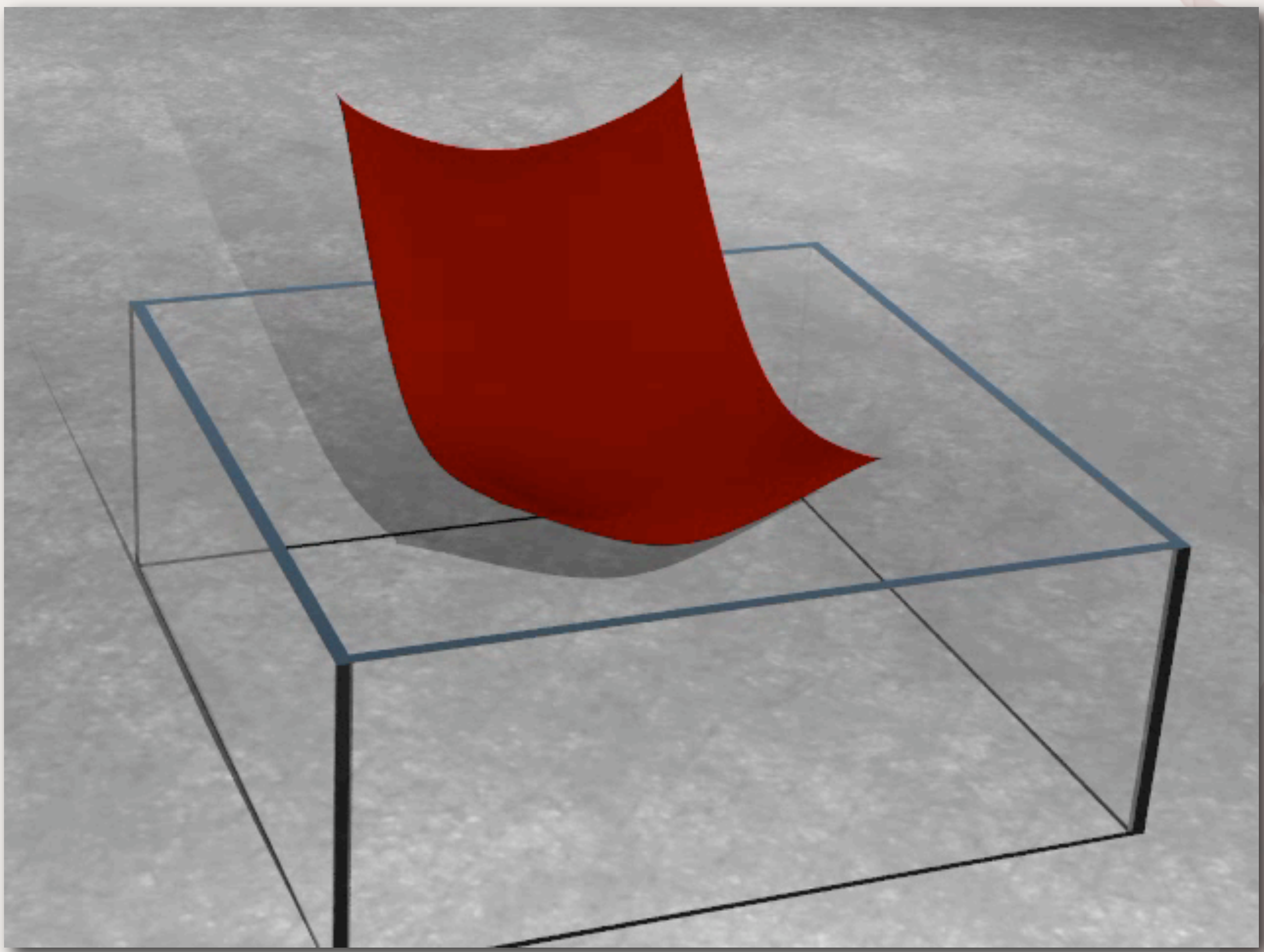

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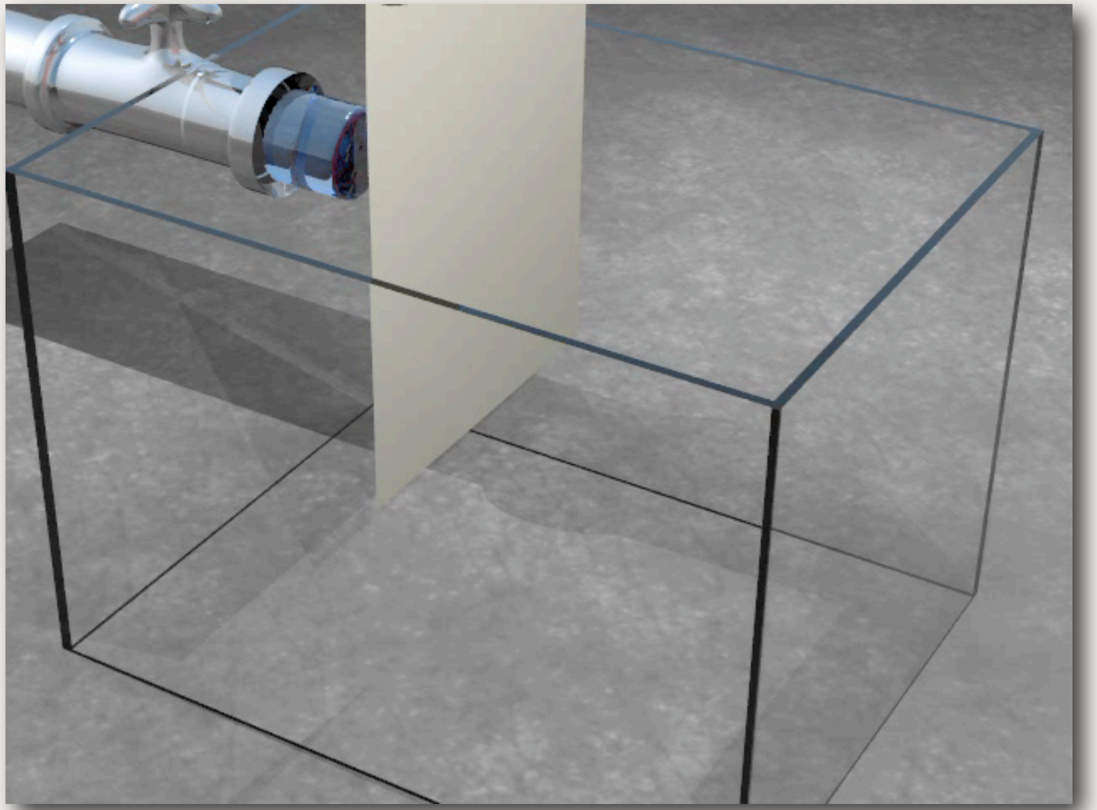

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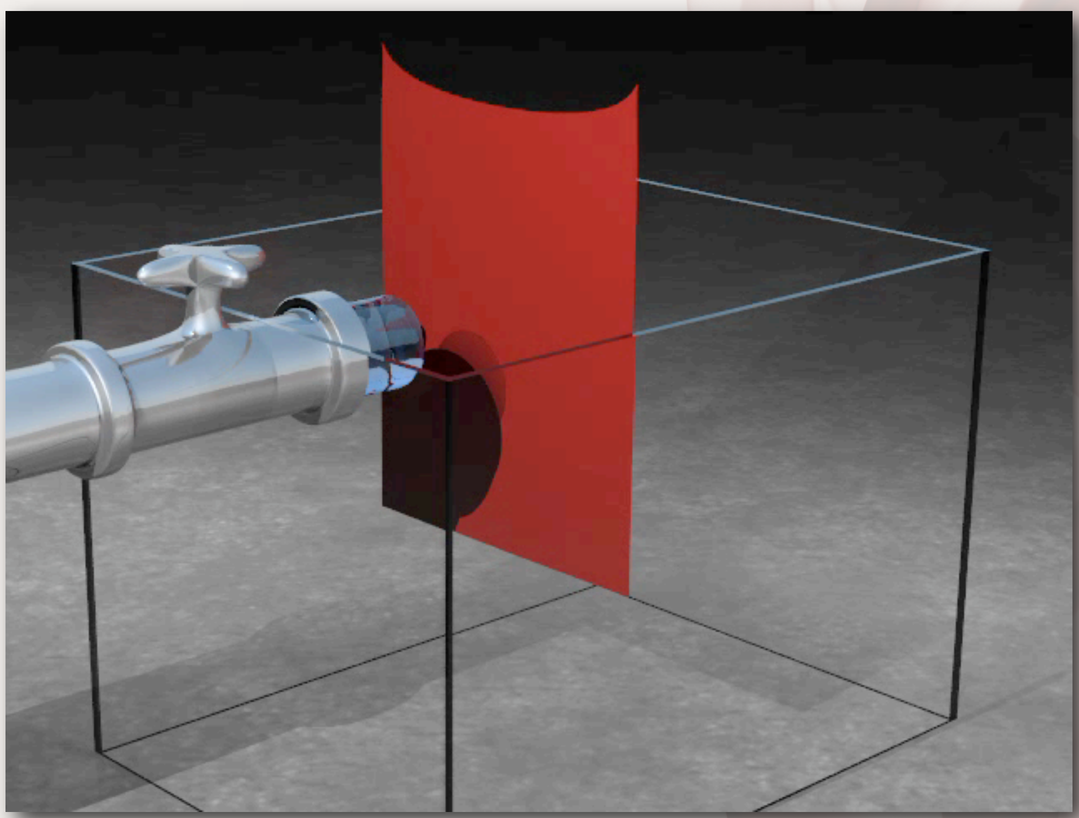

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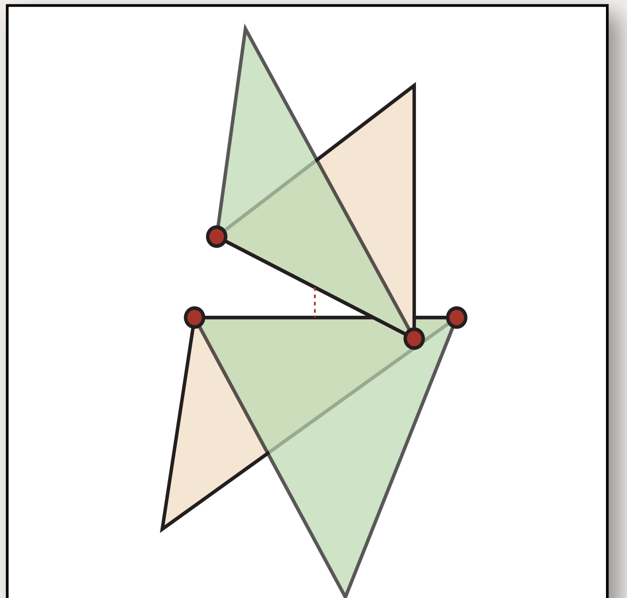
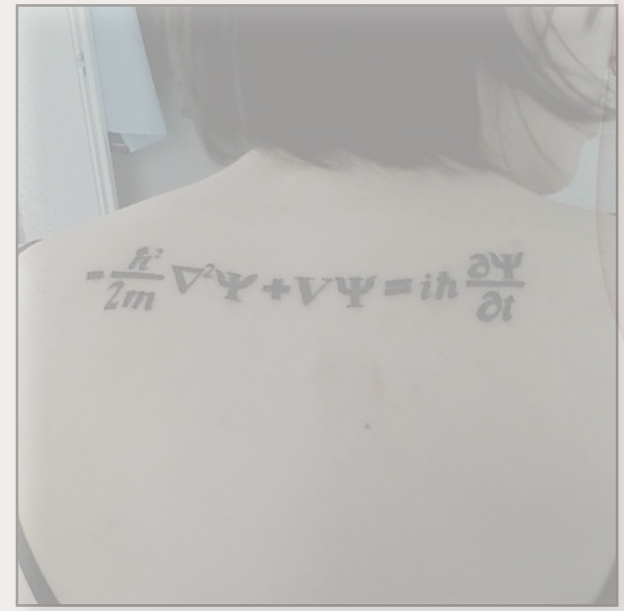
Outline

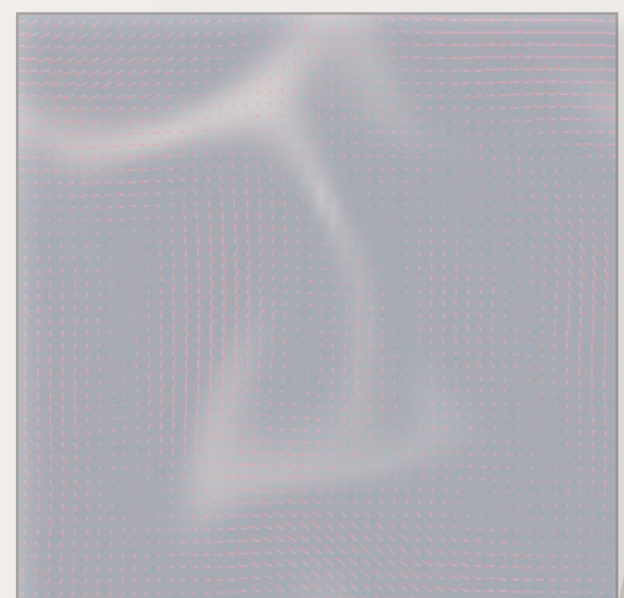


Cloth Collisions



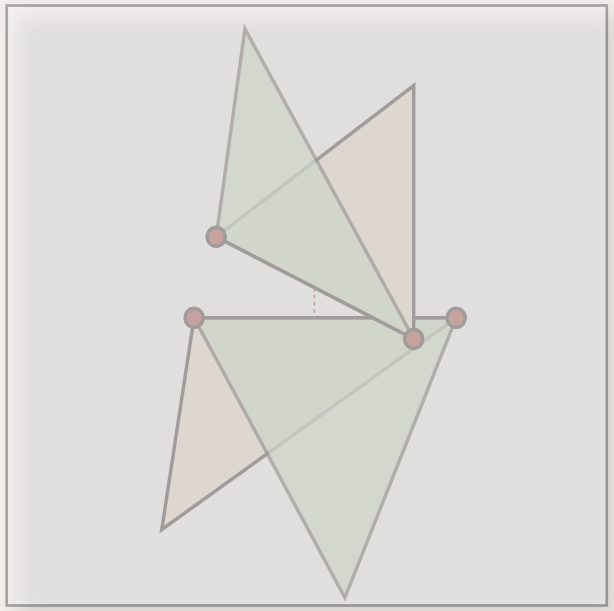
PDEs

Fluid Physics



Fluid Simulation

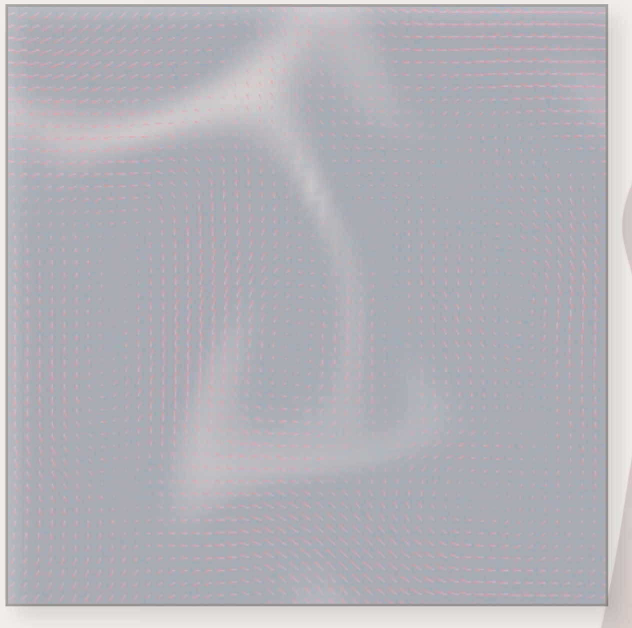
Outline



Cloth Collisions

PDEs

Fluid Physics



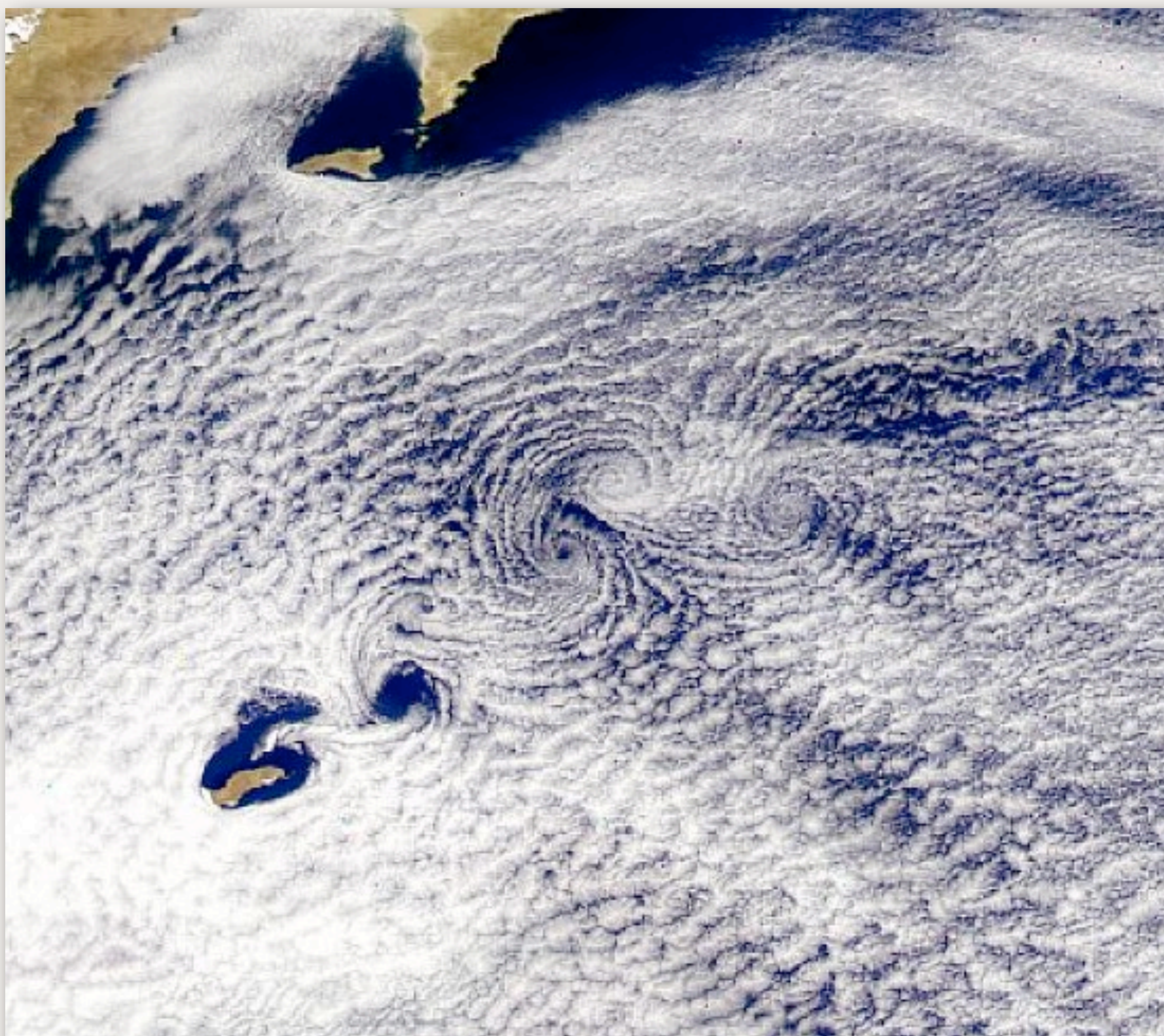
Fluid Simulation

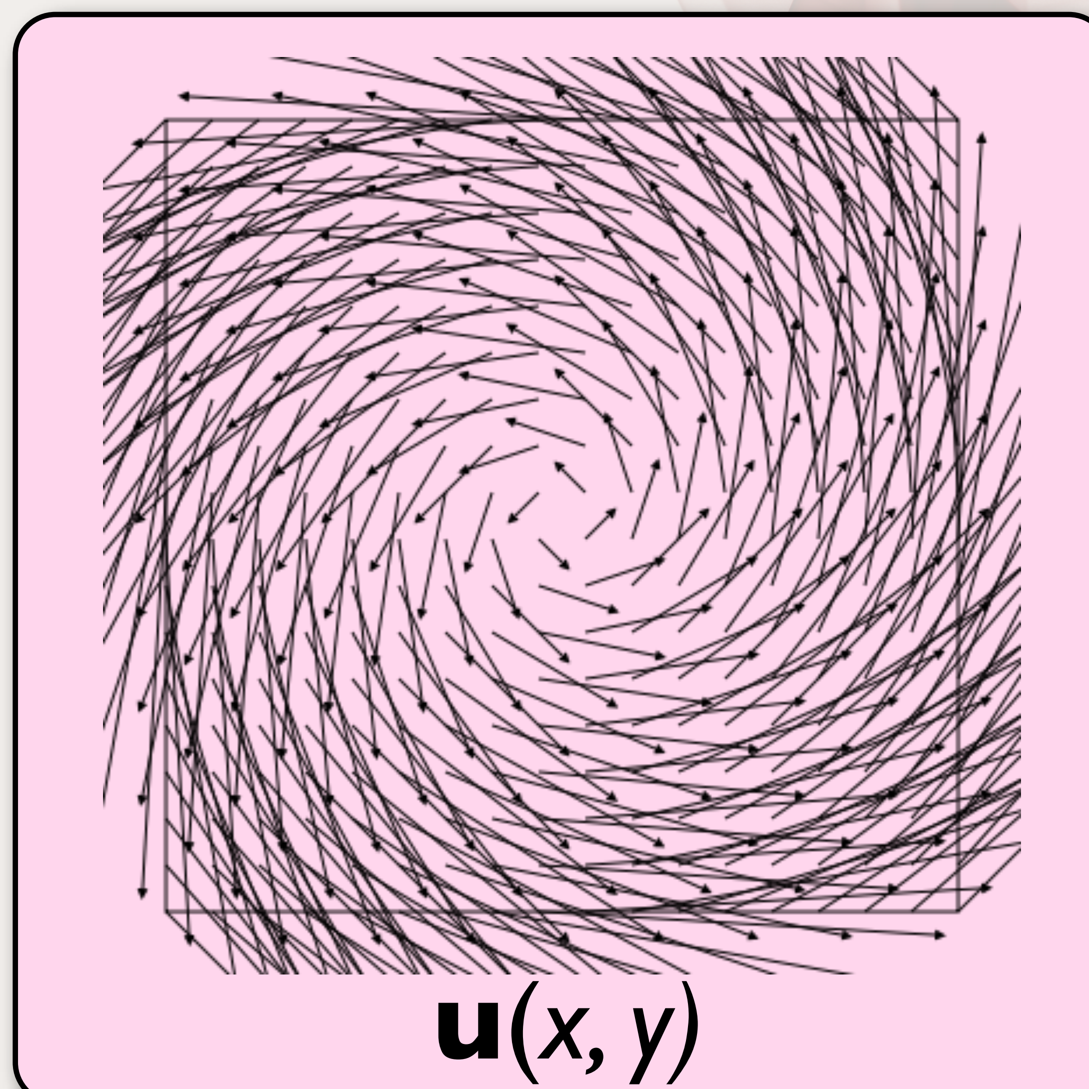
Teaser

making of Audi Commercial - http://motionographer.com/theater/audi-filter-making-of/

Velocity

- How do we represent velocity...
  - As a *function* called a vector field.





Demo

What is a PDE?

- **Ordinary Differential Equation:**

$$\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q})$$

- **Partial Differential Equation:**

$$\dot{q}(x, y) = f\left(q, \frac{\partial q}{\partial x}, \frac{\partial q}{\partial y}, \ldots\right)$$

What is a PDE?

- **Ordinary Differential Equation:**

$$\dot{\mathbf{q}} = \mathbf{f}(\mathbf{q})$$

- **Partial Differential Equation:**

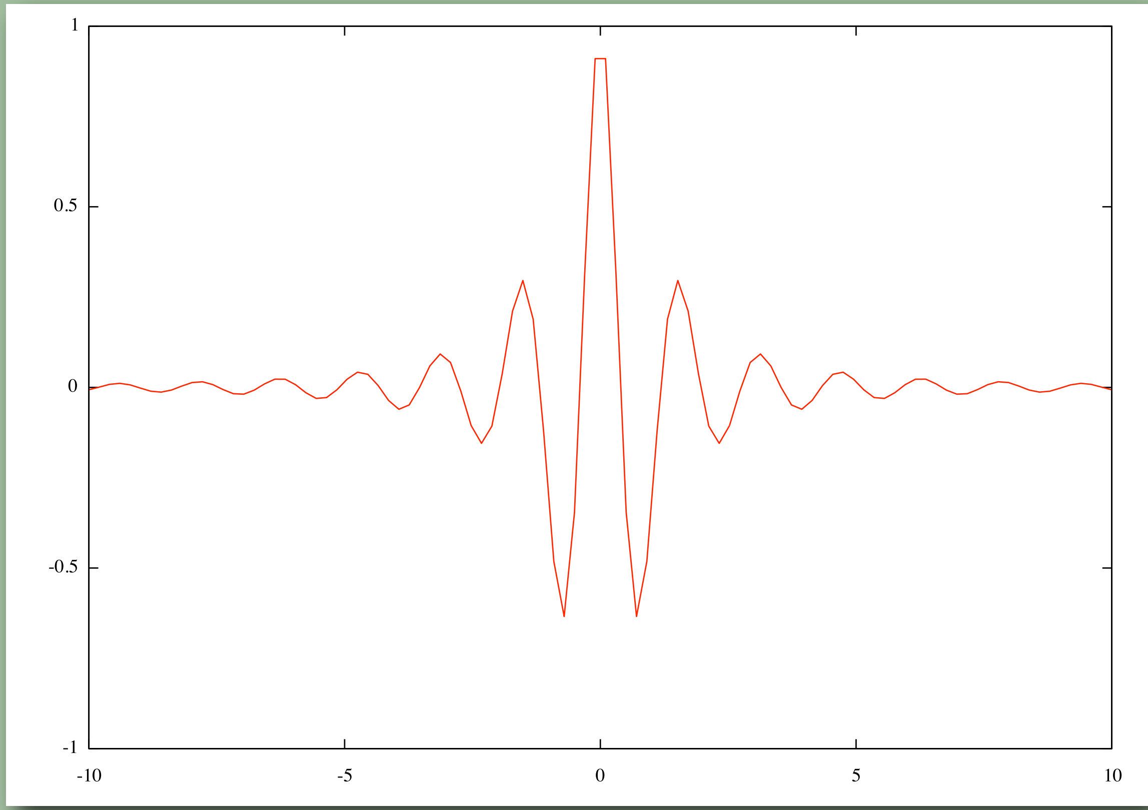
$$\dot{q}(x, y) = f\left(q, \frac{\partial q}{\partial x}, \frac{\partial q}{\partial y}, \ldots\right)$$

Example

$$\dot{f}(x,t) = -\frac{\partial f}{\partial x}$$

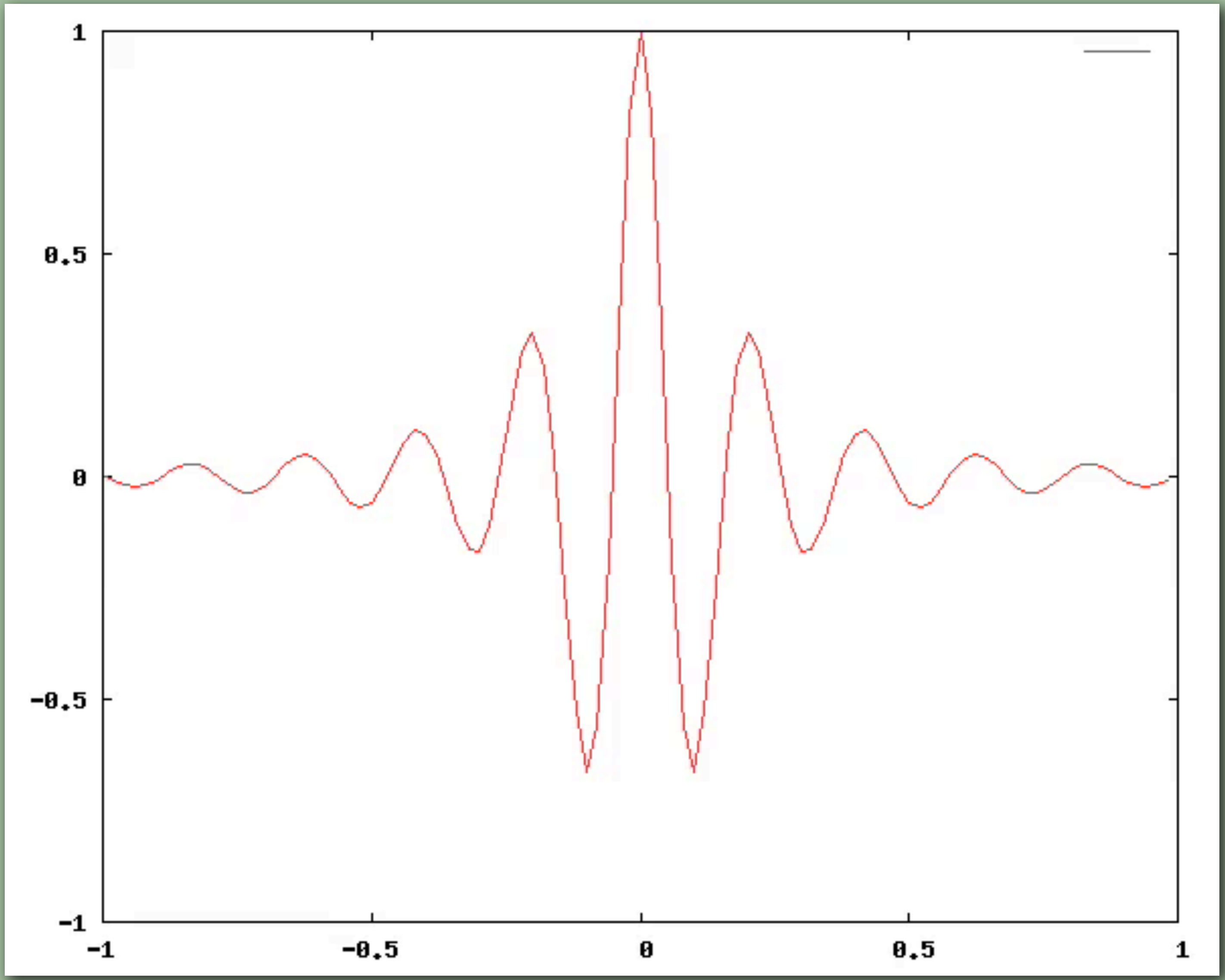
$$f(x,0) = g(x)$$

e.g...

g(x)= 

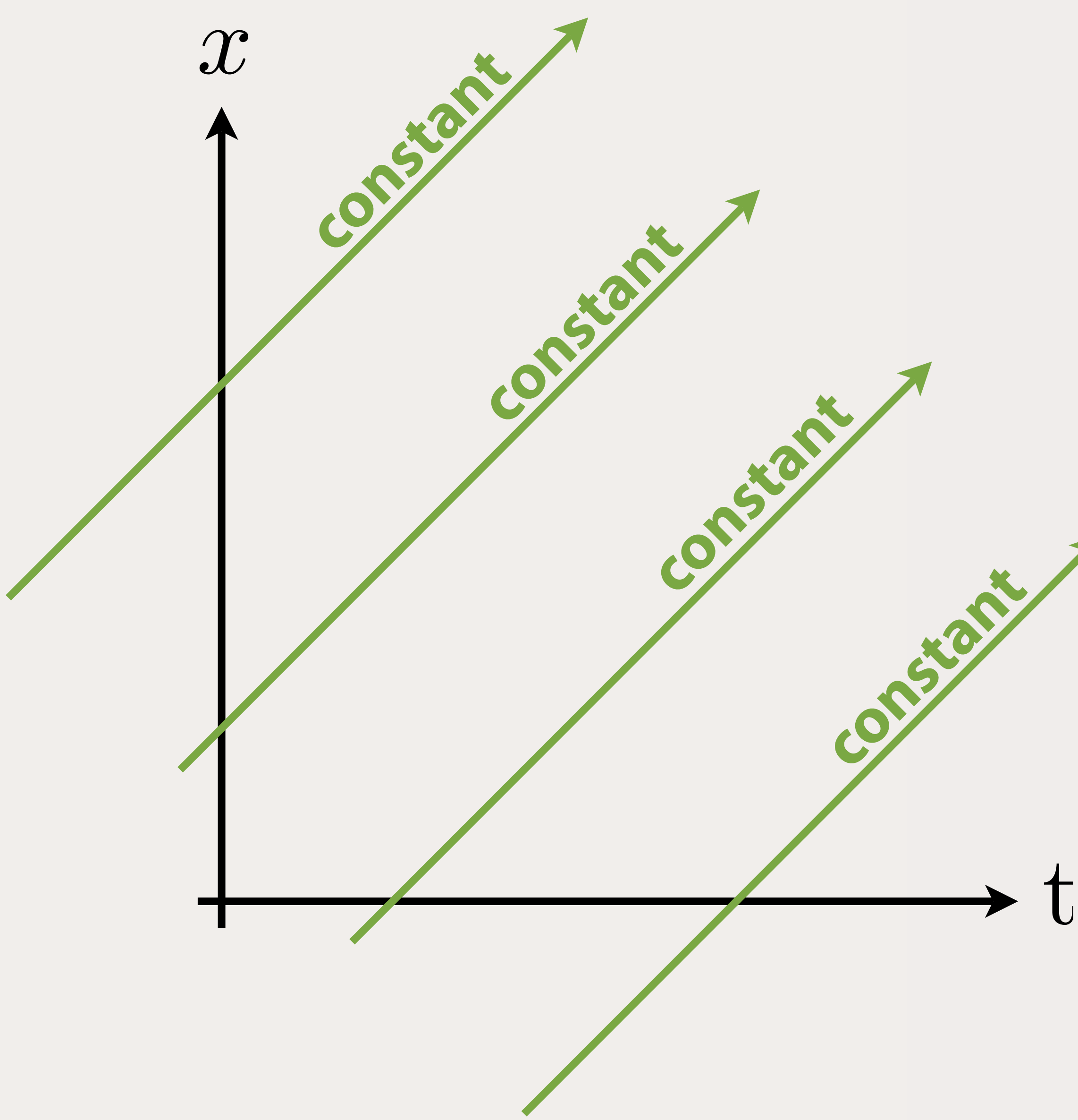
What does this system do?

$$\dot{f}(x,t) = -\frac{\partial f}{\partial x} \quad f(x,0) = g(x)$$

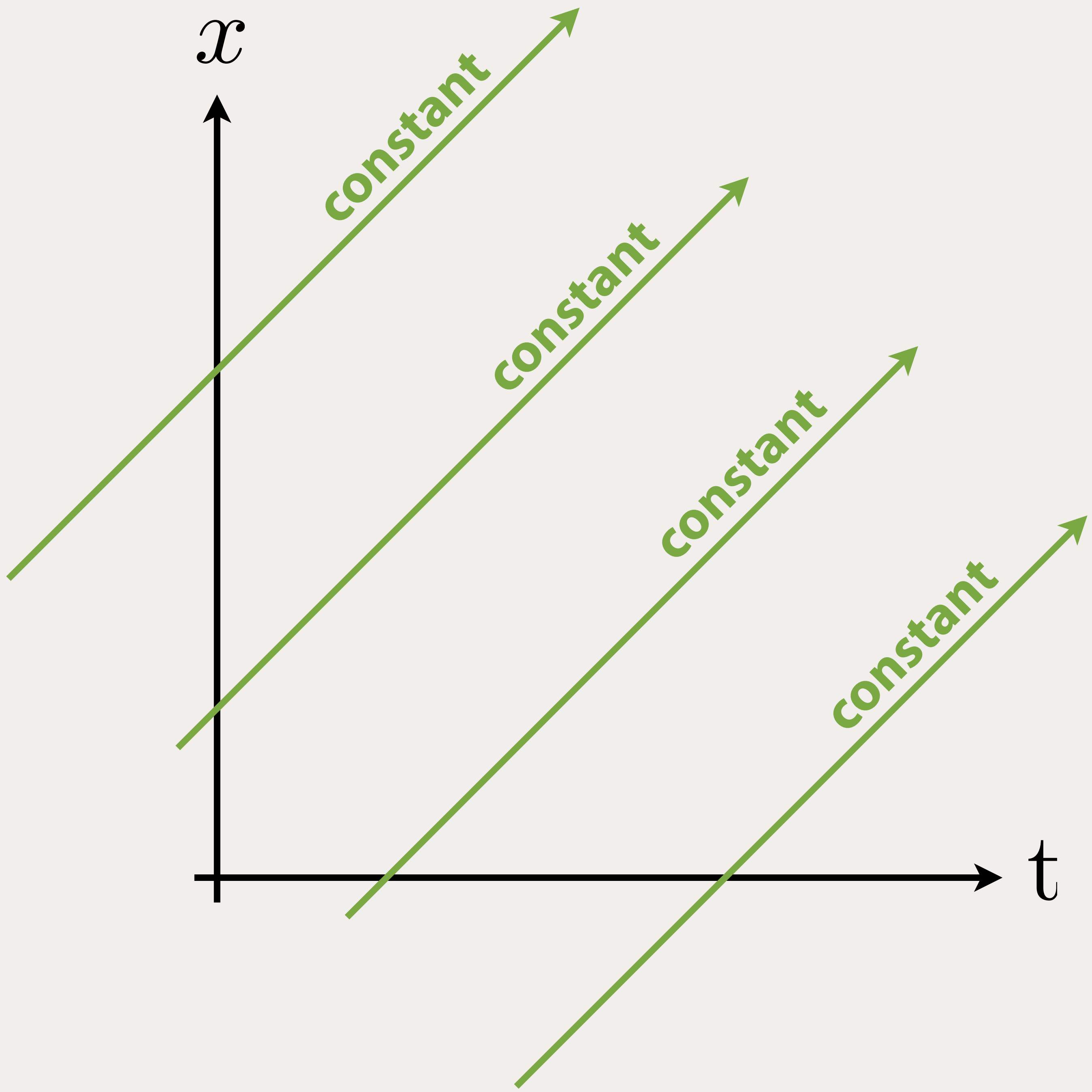


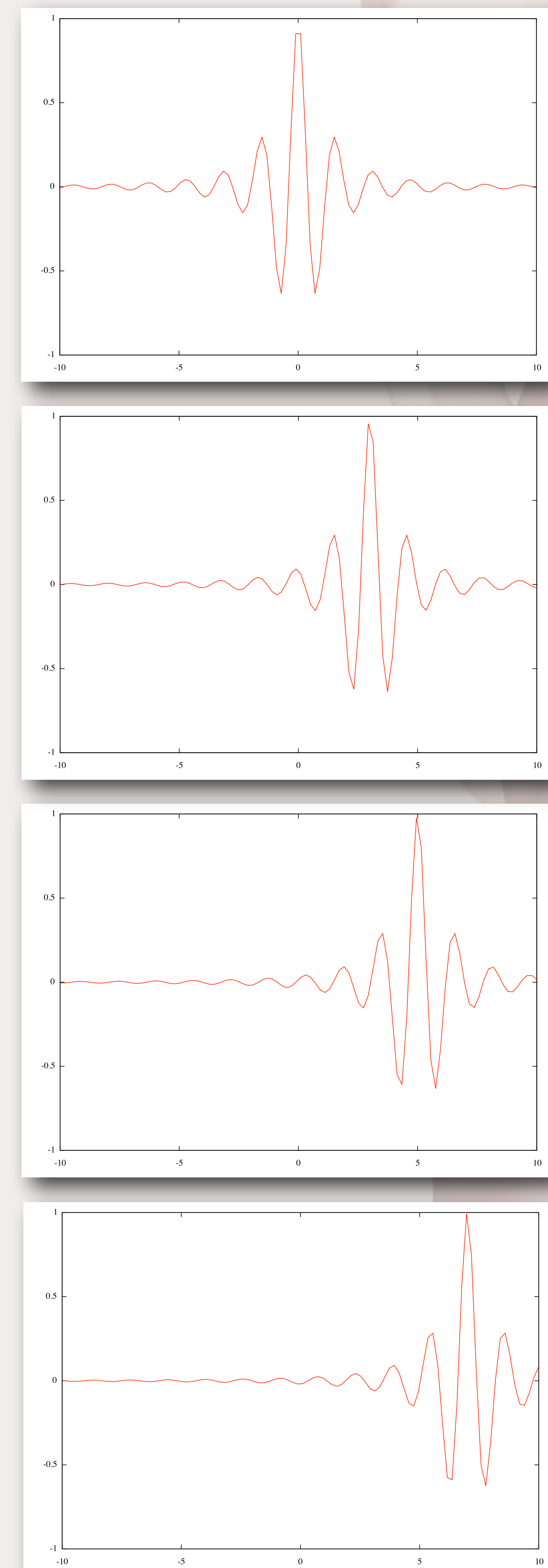
Why?

$$\dot{f}(x,t) = -\frac{\partial f}{\partial x} \;\rightarrow\; [\partial_t f \;\; \partial_x f] \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$



Why?





t=0

t=3

t=5

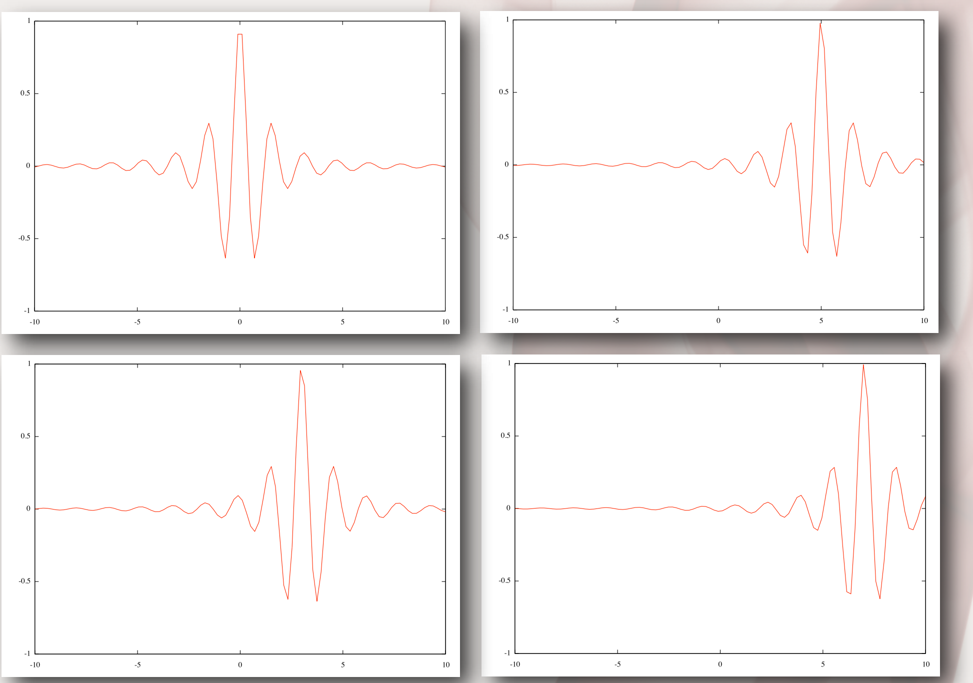
t=7

Closed Form Solution

If...

$$\dot{f}(x,t) = -\frac{\partial f}{\partial x}$$

$$f(x,0) = g(x)$$



then...

$$f(x,t) = g(x-t)$$

Simplified wave propagation.

Numerical Solutions

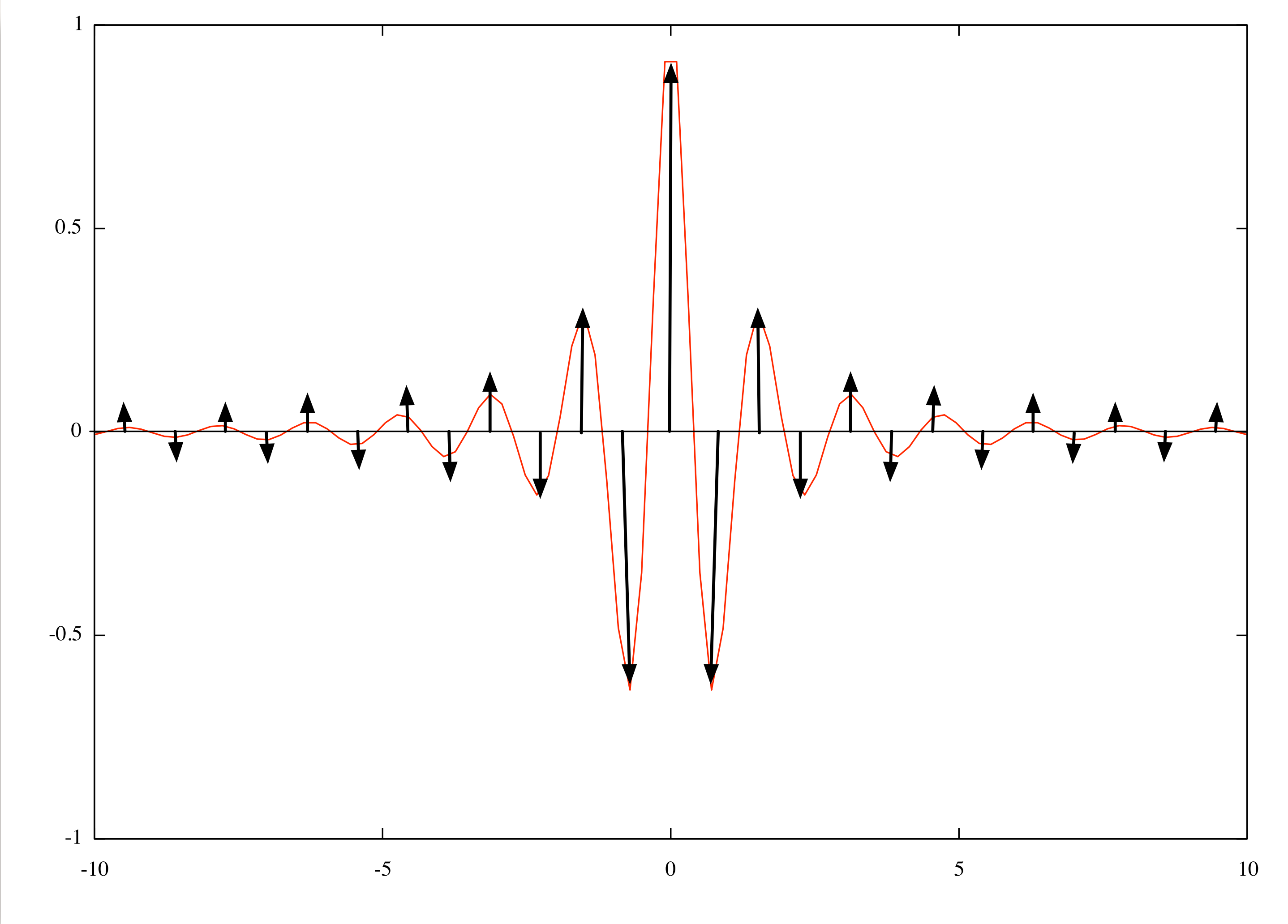
- **How can we solve this numerically?**

$$\dot{f}(x,t) = -\frac{\partial f}{\partial x}$$

- **Answer:**
  - ***First discretize in space.***
  - ***Then discretize in time.***

Discretize in Space

- **Turn our PDE into an ODE...**

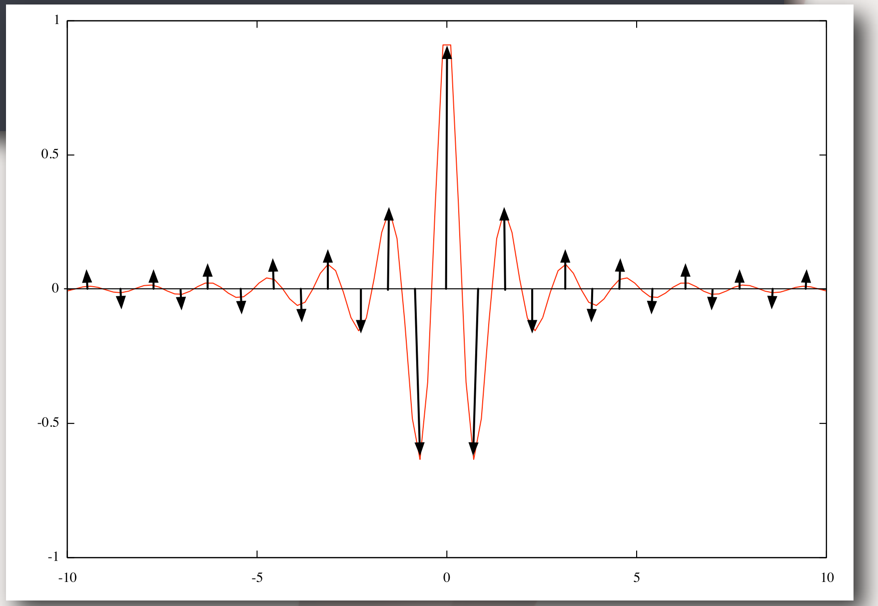


***f* becomes a "discrete function of space"...**

space between spikes = Δx

Data Structures

```
class Function1D {
 public:
  int len() const;
  double set(int index, double value);
  double get(int index) const;
  double getDeriv(int index) const;
};
```



Computing Derivatives

- **Now that we have a discrete function:**

$$\dot{f}_i = -\frac{\partial f}{\partial x}$$

- **What do we do about the derivative:**

$$\dot{f}_i = -\left(\frac{f_{i+1} - f_i}{\Delta x}\right)$$ **Forward Differencing**

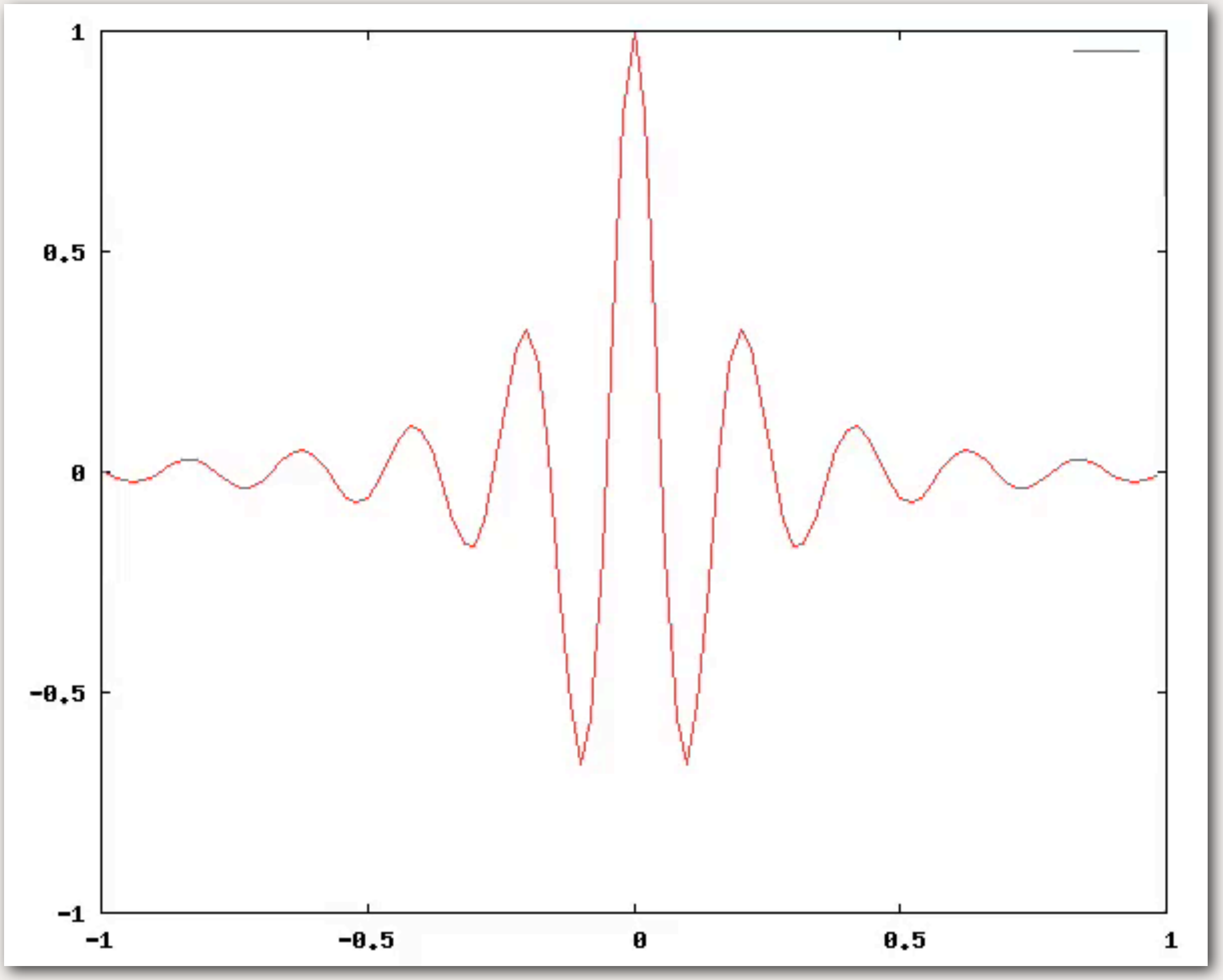
$$\dot{f}_i = -\left(\frac{f_i - f_{i-1}}{\Delta x}\right)$$ **Backward Differencing**

$$\dot{f}_i = -\left(\frac{f_{i+1} - f_{i-1}}{2\Delta x}\right)$$ **Central Differencing**

Discretize in Time

Euler's method backward differencing:

$$f_i^{t+1} = f_i^t - \Delta t \left(\frac{f_i - f_{i-1}}{\Delta x} \right)$$



Δt = 0.01

Discretize in Time

Euler's method backward differencing:

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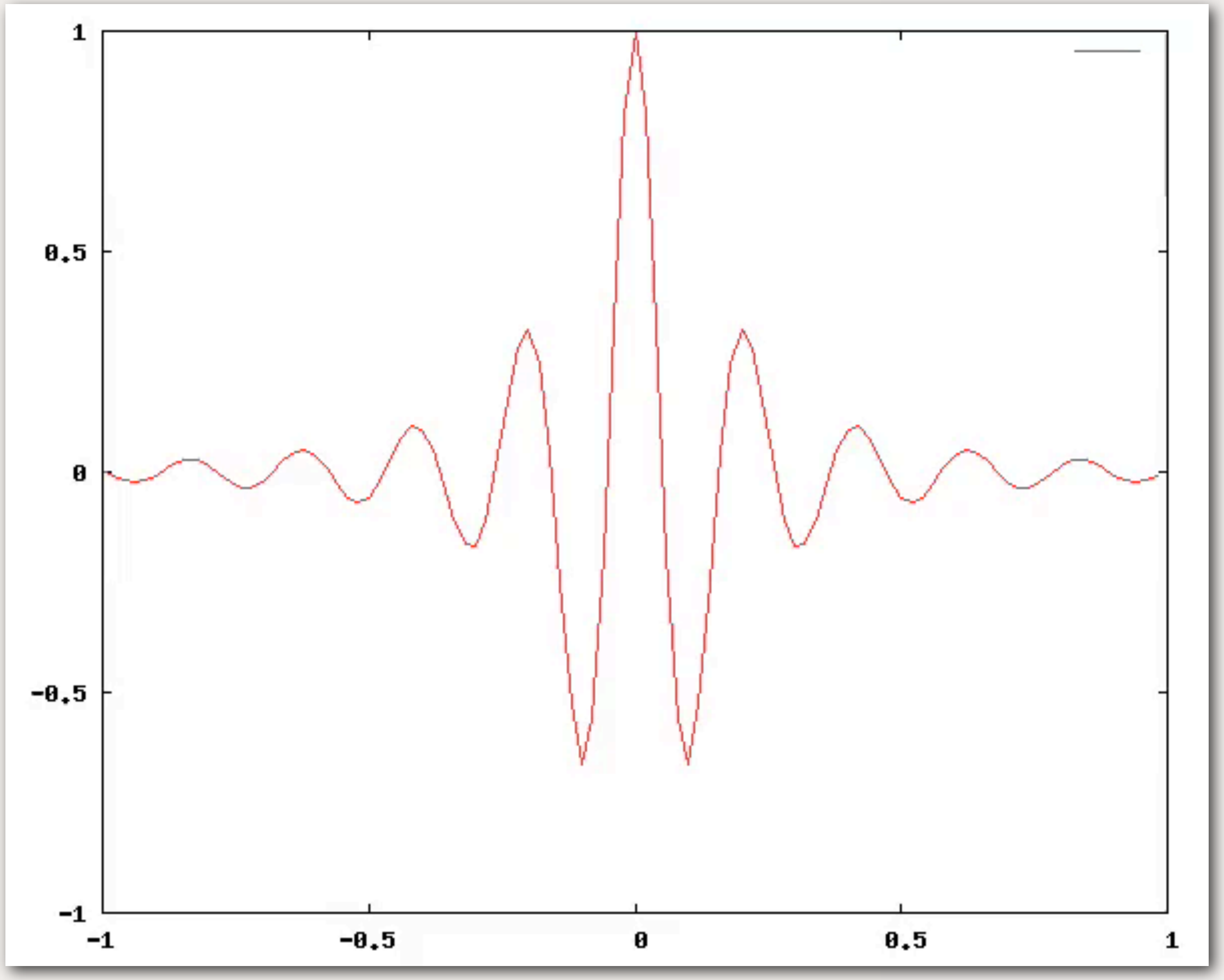
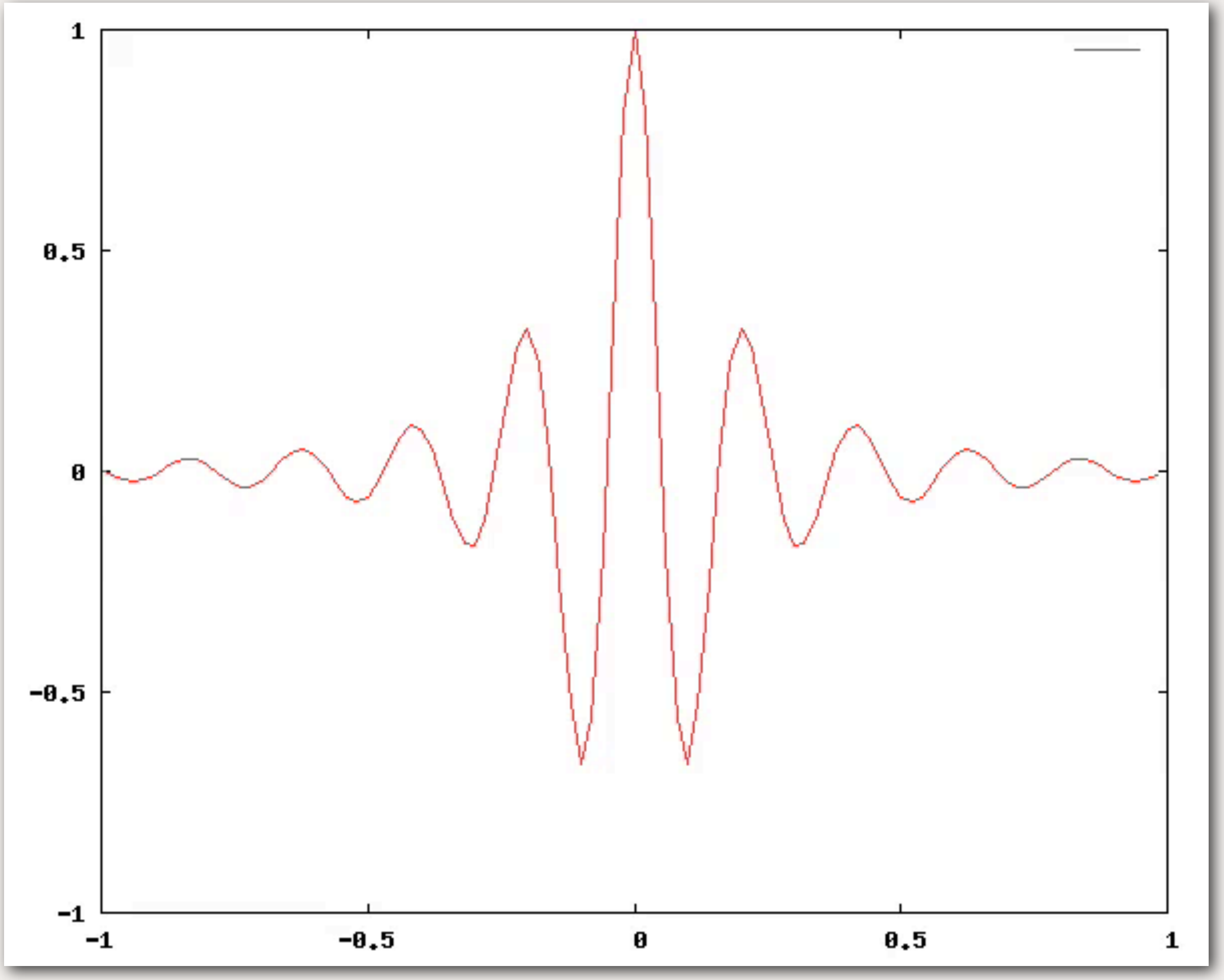


Δt = 0.1

Discretize in Time

Euler's method backward differencing:

$$f_i^{t+1} = f_i^t - \Delta t \left(\frac{f_i - f_{i-1}}{\Delta x} \right)$$



Δt = 1.0

Discretize in Time

Euler's method backward differencing:

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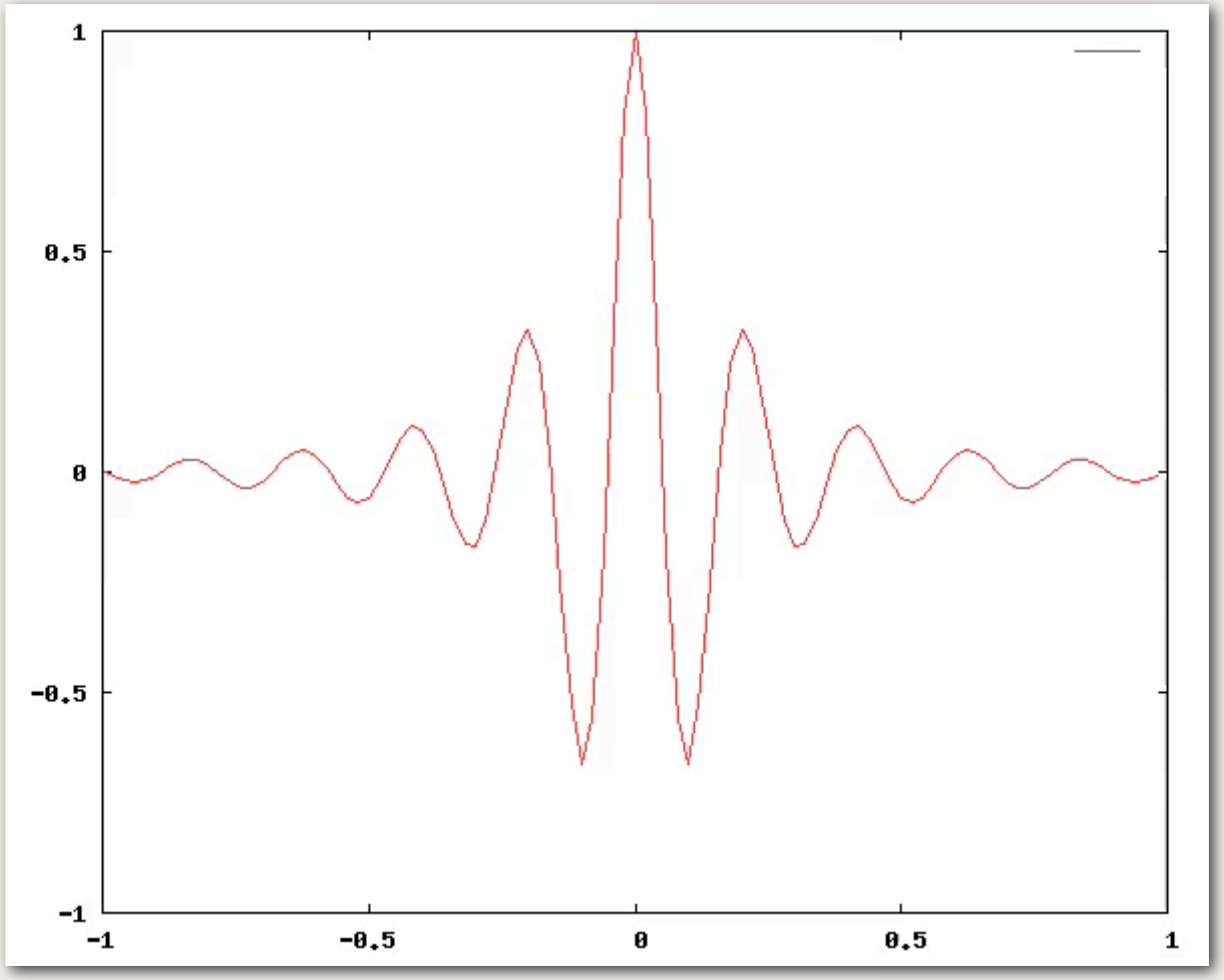
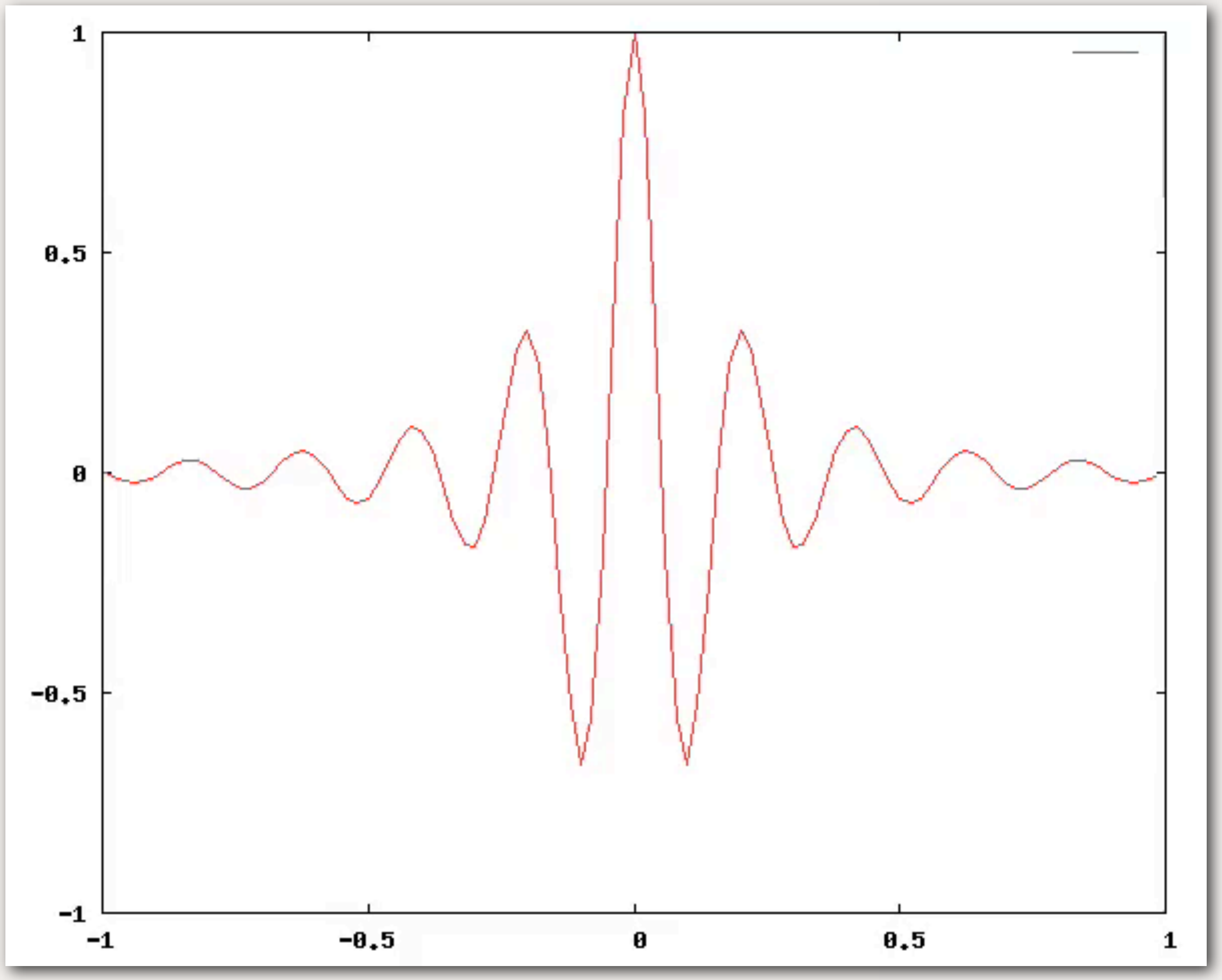


Δt = 2.0

Discretize in Time

Euler's method forward differencing:

$$f_i^{t+1} = f_i^t - \Delta t \left(\frac{f_{i+1} - f_i}{\Delta x} \right)$$

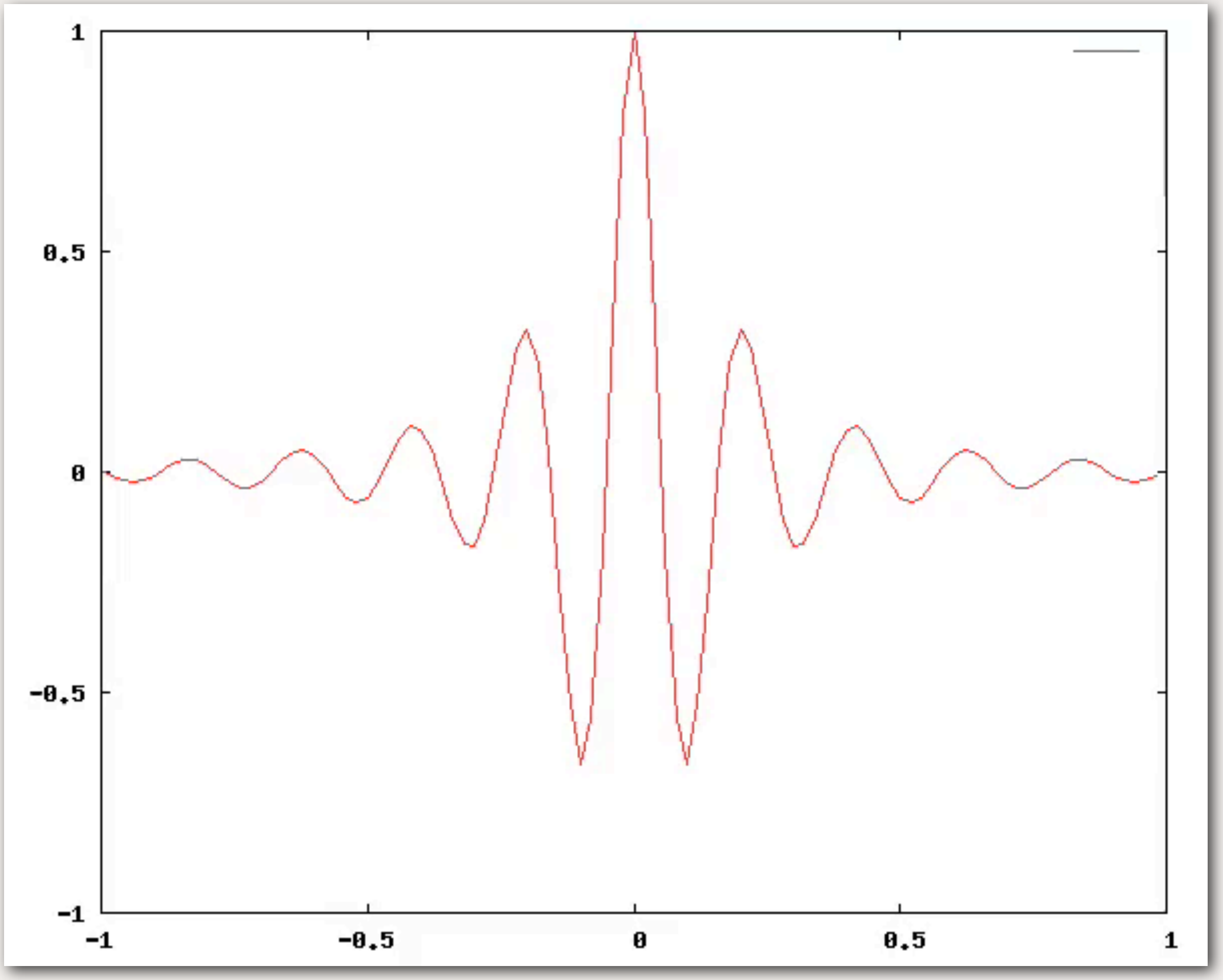


Δt = 2.0

Discretize in Time

Euler's method forward differencing:

$$f_i^{t+1} = f_i^t - \Delta t \left(\frac{f_{i+1} - f_i}{\Delta x} \right)$$

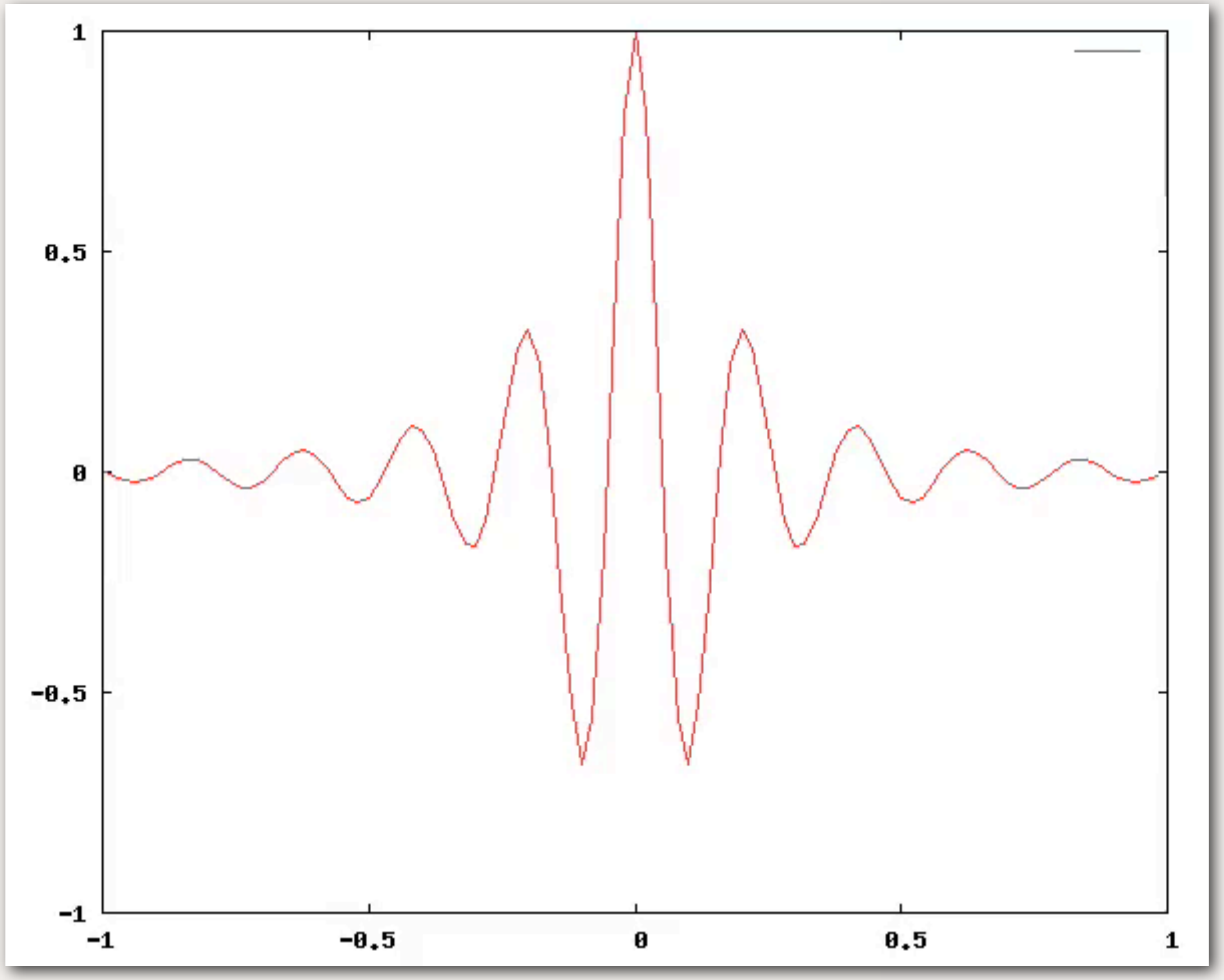


Δt = 1.0

Discretize in Time

Euler's method forward differencing:

$$f_i^{t+1} = f_i^t - \Delta t \left(\frac{f_{i+1} - f_i}{\Delta x} \right)$$

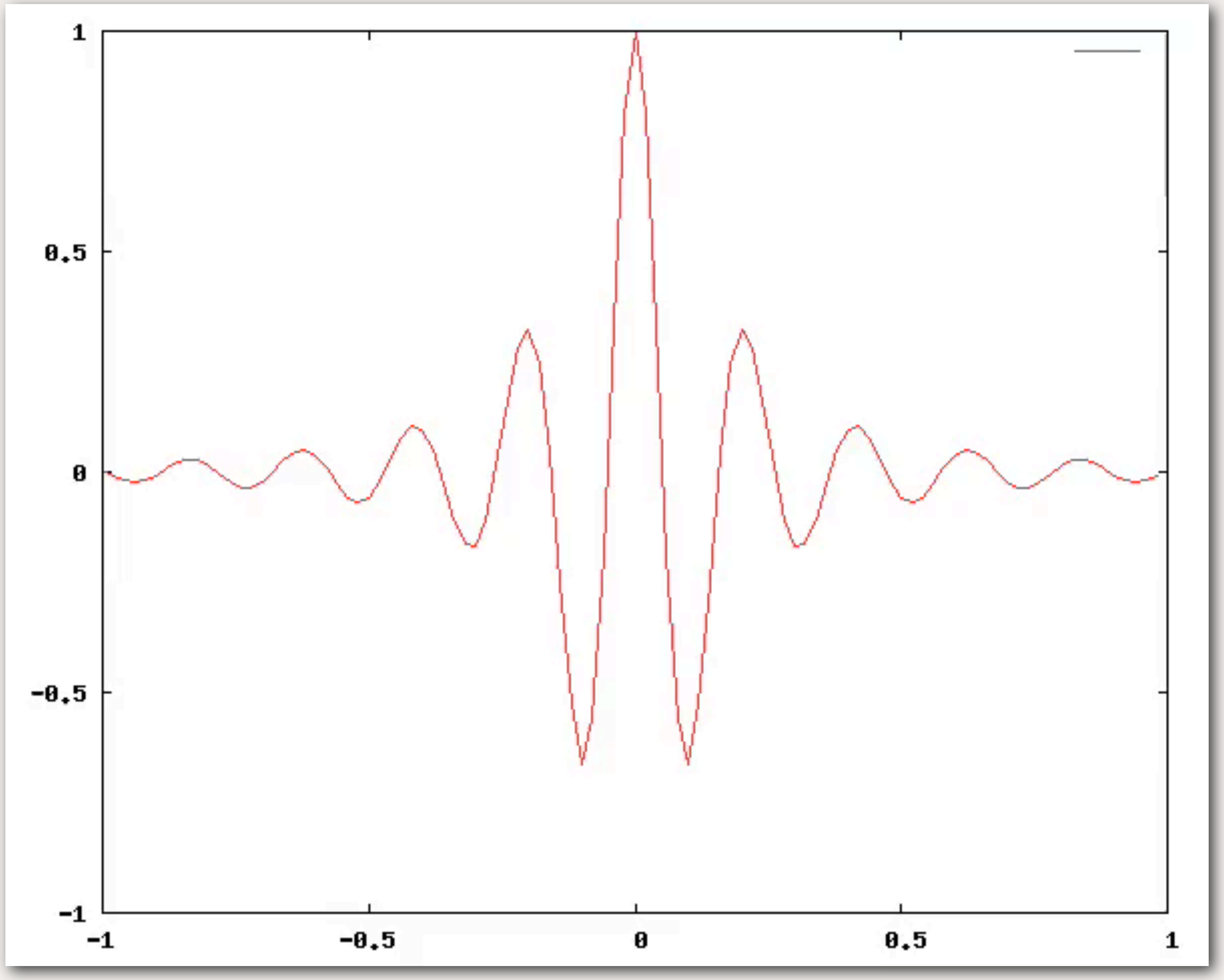


Δt = 0.1

Discretize in Time

Euler's method forward differencing:

$$f_i^{t+1} = f_i^t - \Delta t \left(\frac{f_{i+1} - f_i}{\Delta x} \right)$$



Δt = 0.01

In short.

$$\dot{f}(x,t) = -\frac{\partial f}{\partial x}$$

dt=	0.01	0.1	1.0	2.0
Backward	Diffusive	Diffusive	Perfect?	Crap.
Forward	Crap.	Crap.	Crap.	Crap.

Why?

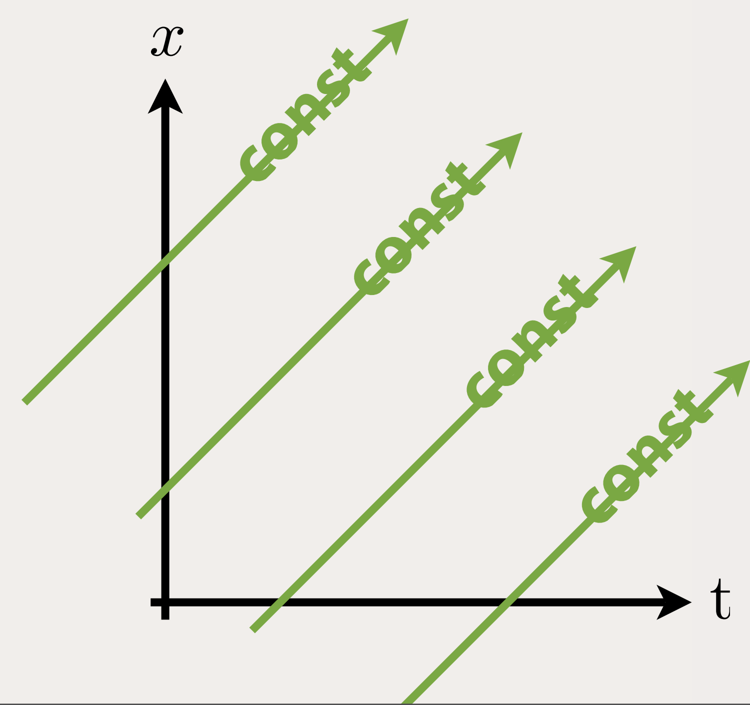
Can we do better?

- **Recall...**

$$f(x,t) = g(x-t)$$

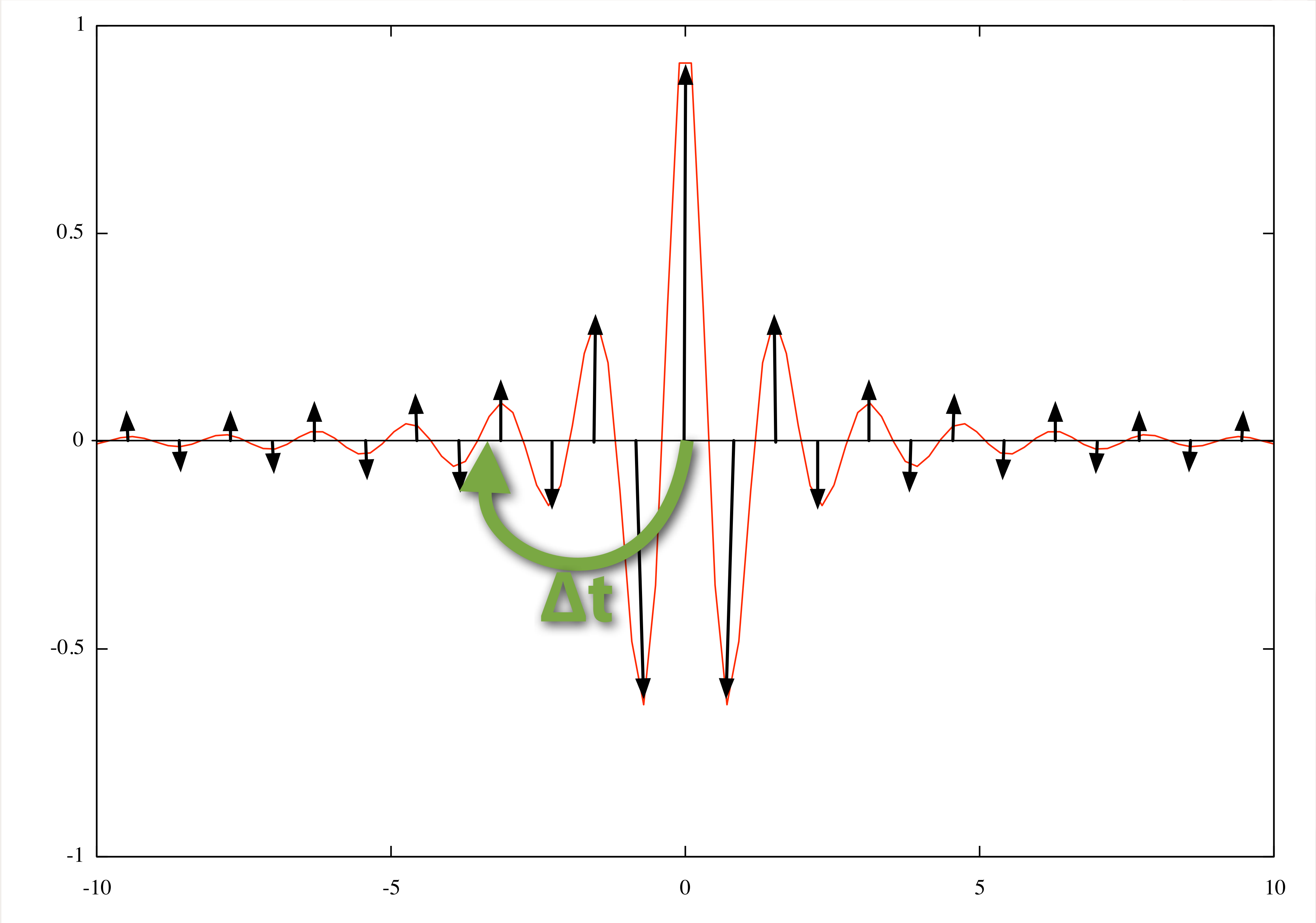
- **Information propagates "to the right"**

$$f(x, t+\Delta t) = f(x-\Delta t, t)$$



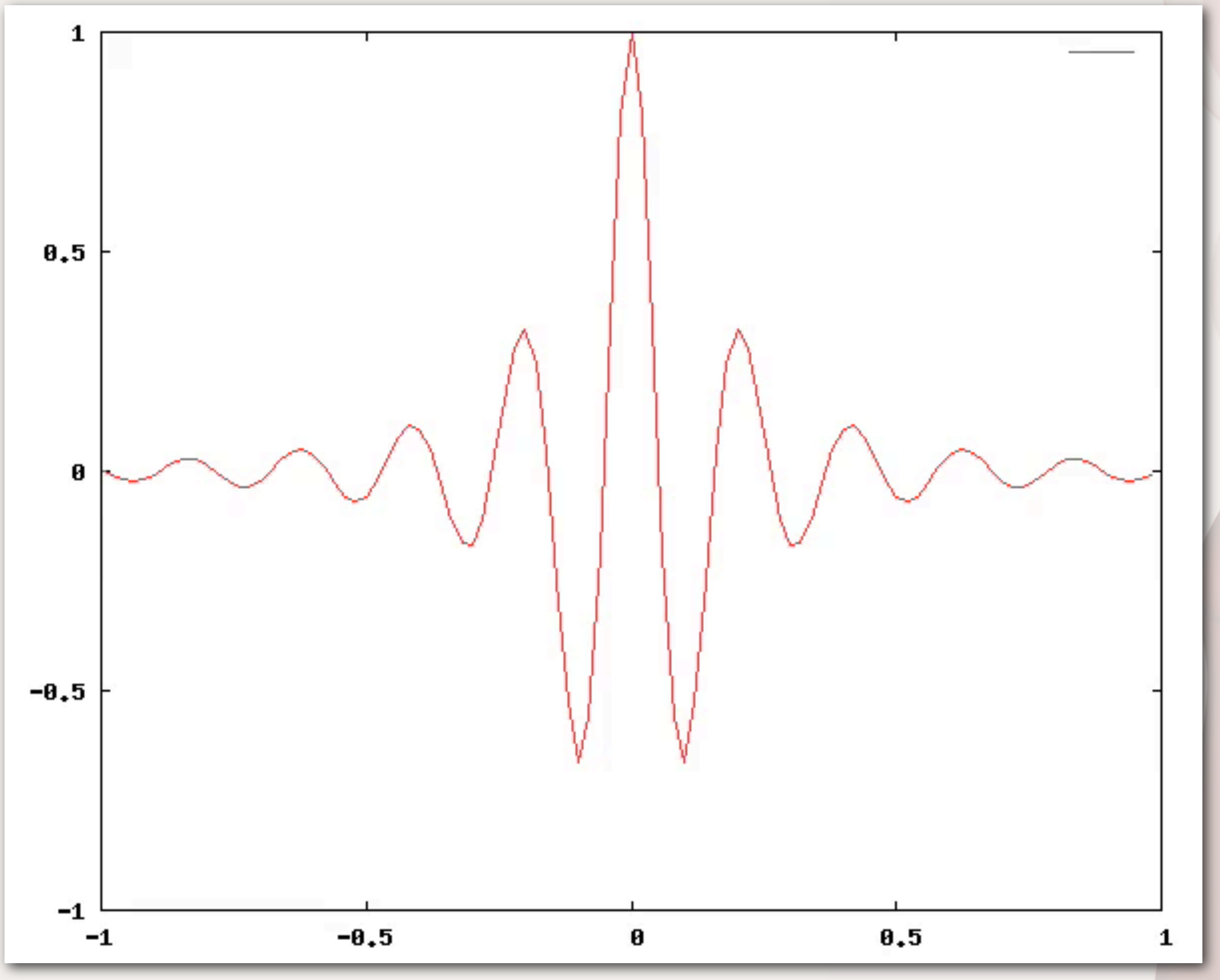
Semi-Lagrangian

$$f(x, t + \Delta t) = f(x - \Delta t, t)$$

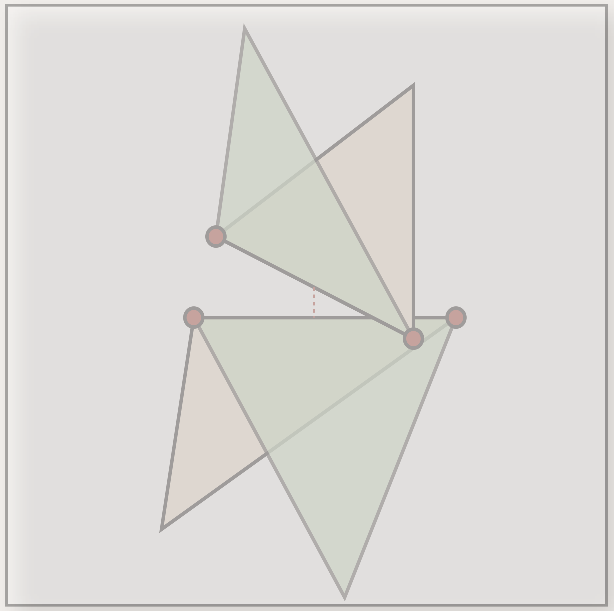


Semi-Lagrangian

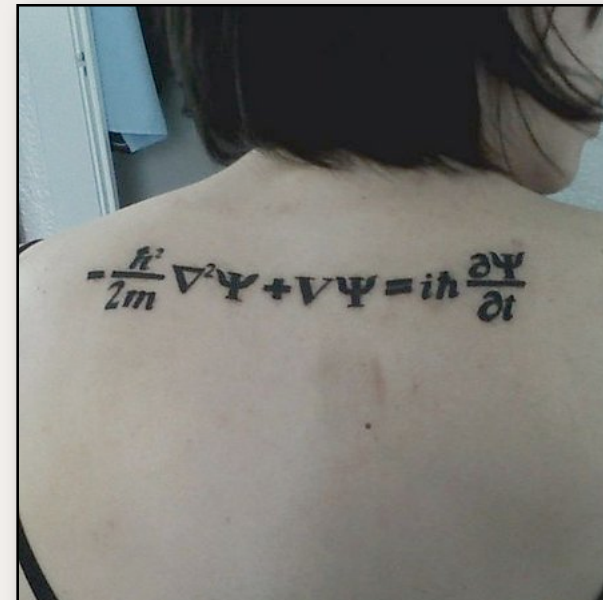
$$f(x, t + \Delta t) = f(x - \Delta t, t)$$



Outline

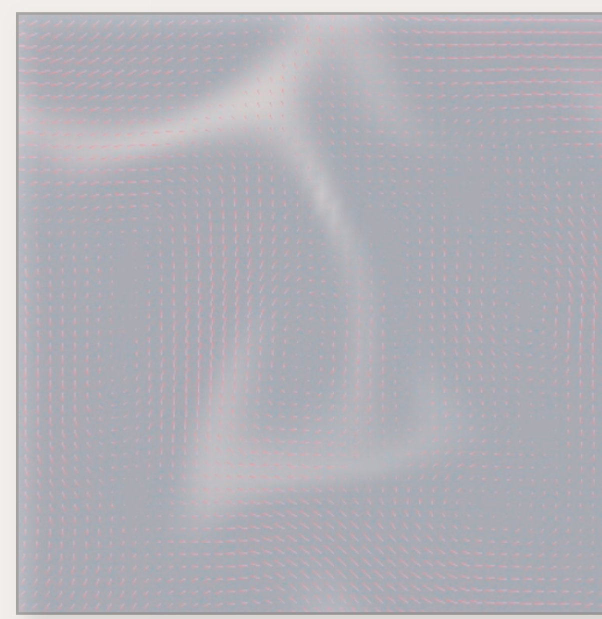


Cloth Collisions



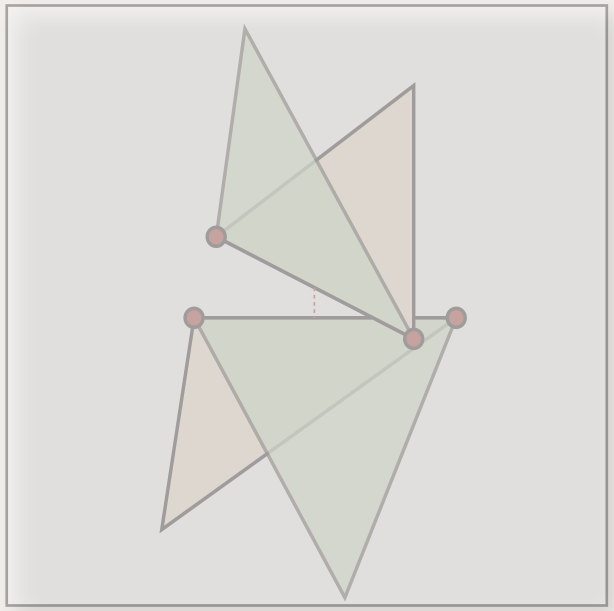
PDEs

Fluid Physics



Fluid Simulation

Outline



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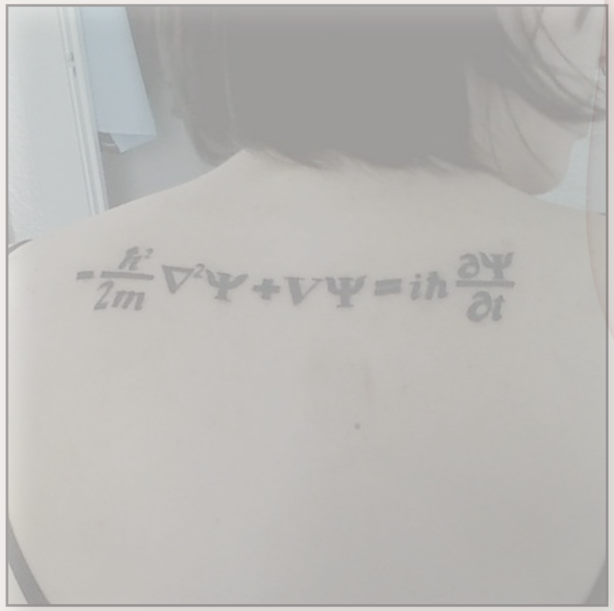
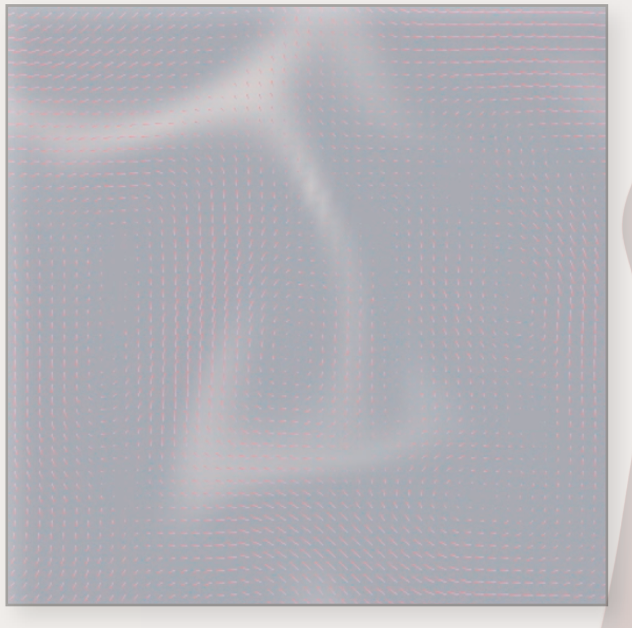


PDEs

Fluid Physics



Fluid Simulation

Incompressible Navier-Stokes Equations

$\rho : \Omega \to [0, 1]$
(density)

Ω
(domain)

$\mathbf{u} : \Omega \to \mathbf{R}^3$
(velocity)

Navier Stokes

- **Density**

$$\frac{\partial \rho}{\partial t} = -(\mathbf{u} \cdot \nabla)\rho$$

- **Velocity**

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

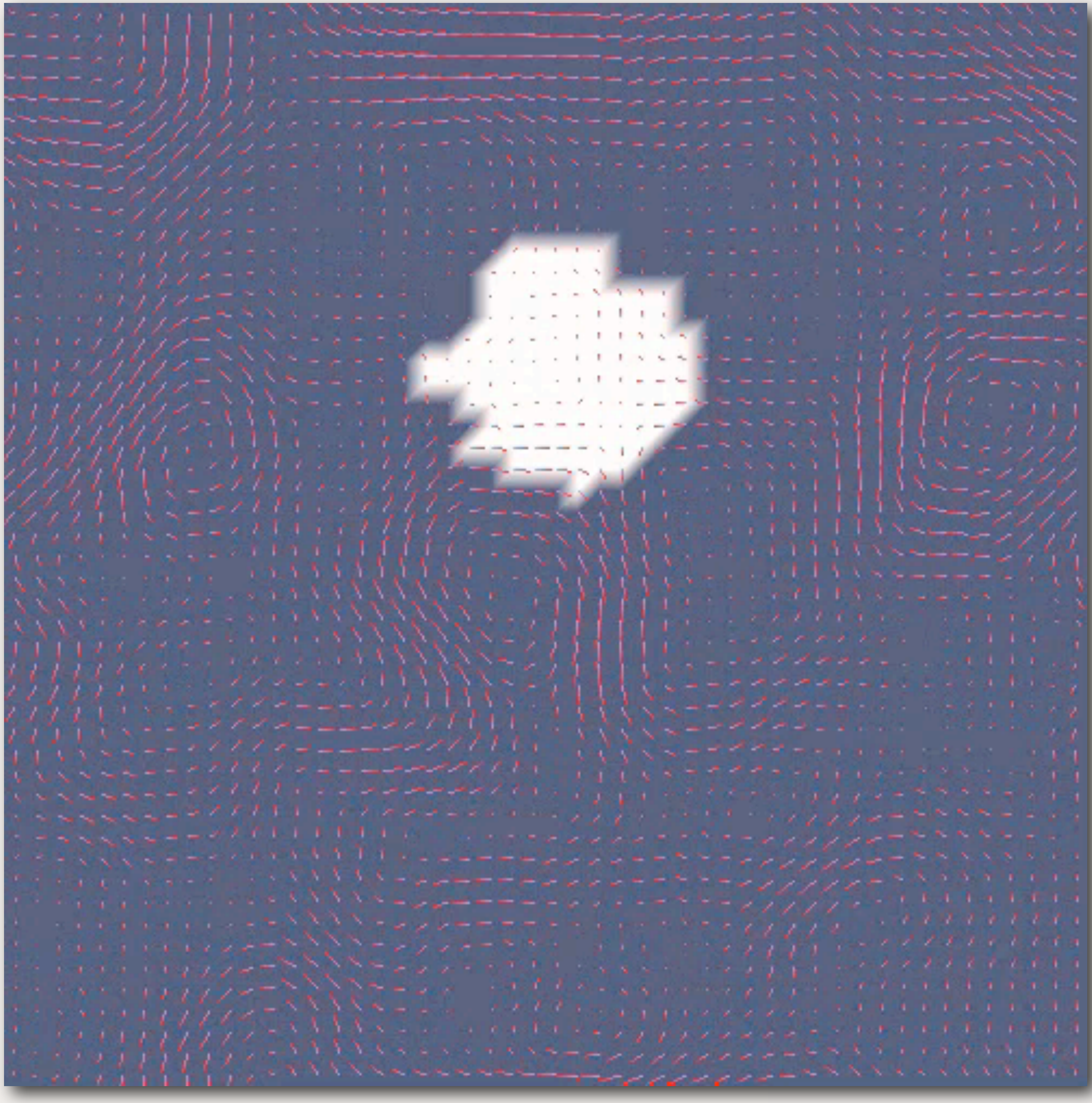
s.t. $\nabla \cdot \mathbf{u} = 0$

Demo

Density Advection

$$\frac{\partial \rho}{\partial t} = -(\mathbf{u} \cdot \nabla)\rho$$

Density Advection

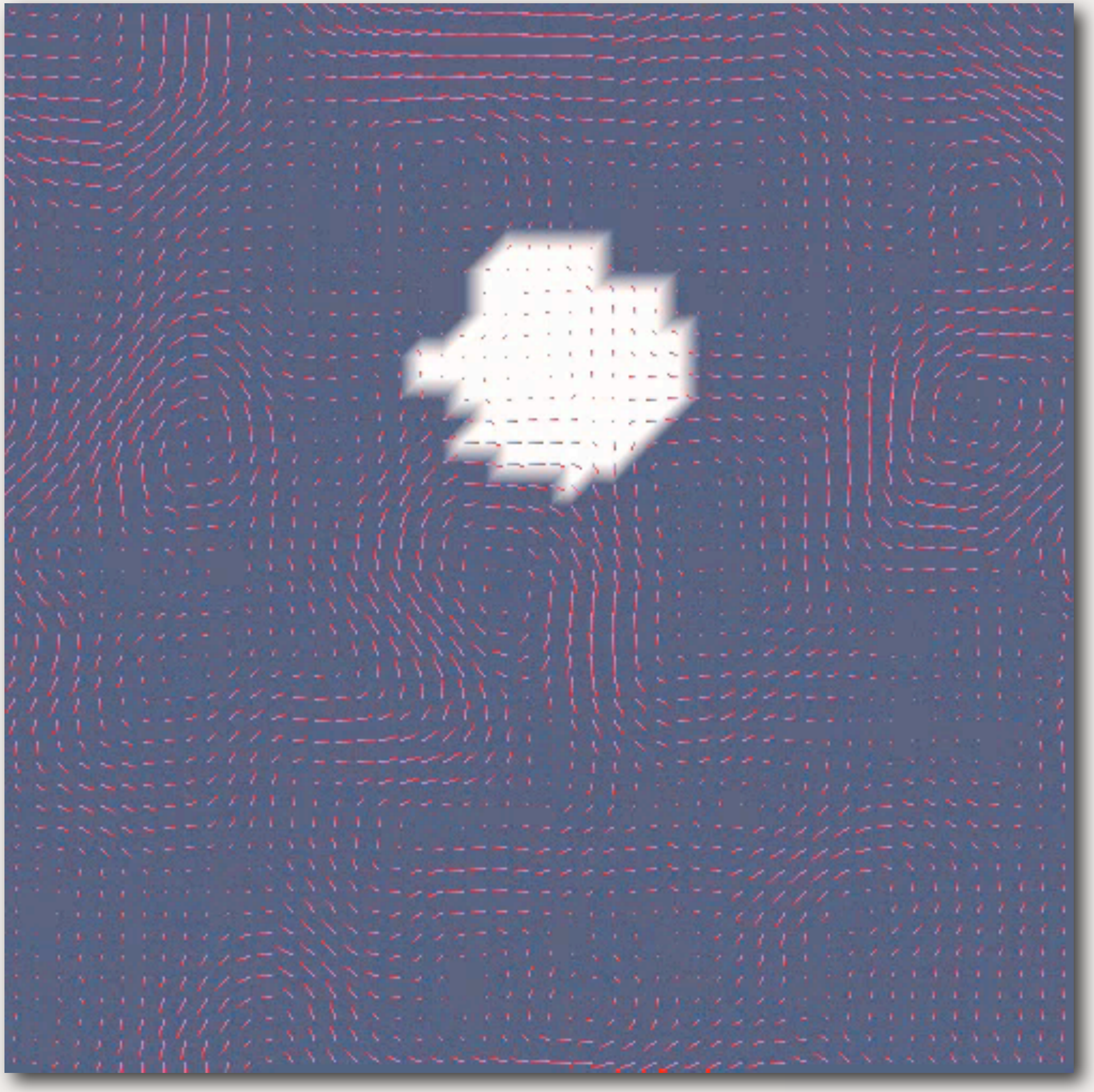


Velocity Advection

$$\frac{\partial \mathbf{u}}{\partial t} = \boxed{-(\mathbf{u}\cdot\nabla)}\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

s.t. $\nabla \cdot \mathbf{u} = 0$

Velocity Advection



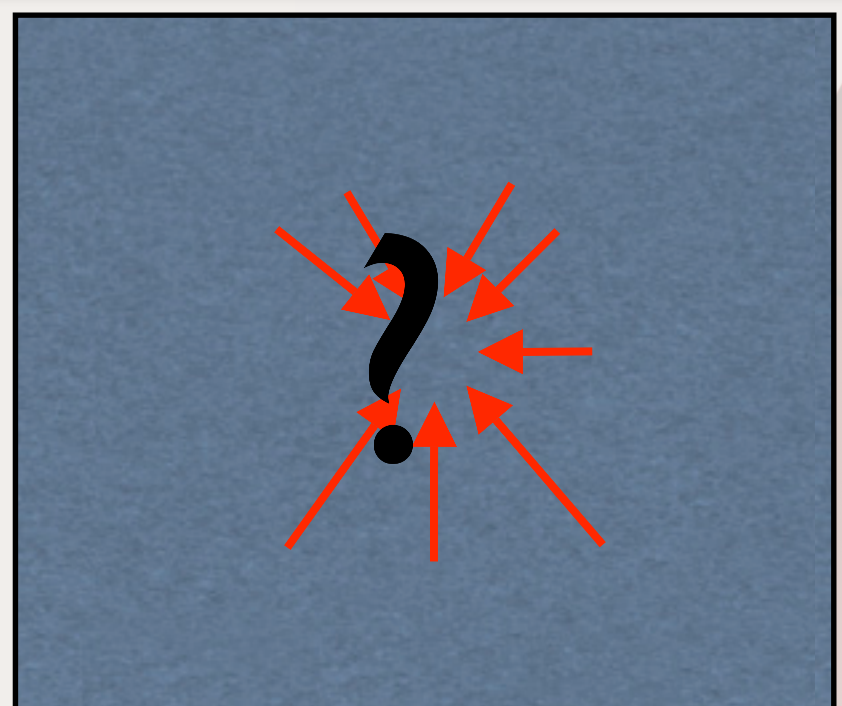
Projection

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \boxed{\frac{1}{r}\nabla p} + s\nabla^2\mathbf{u} + \mathbf{f}$$

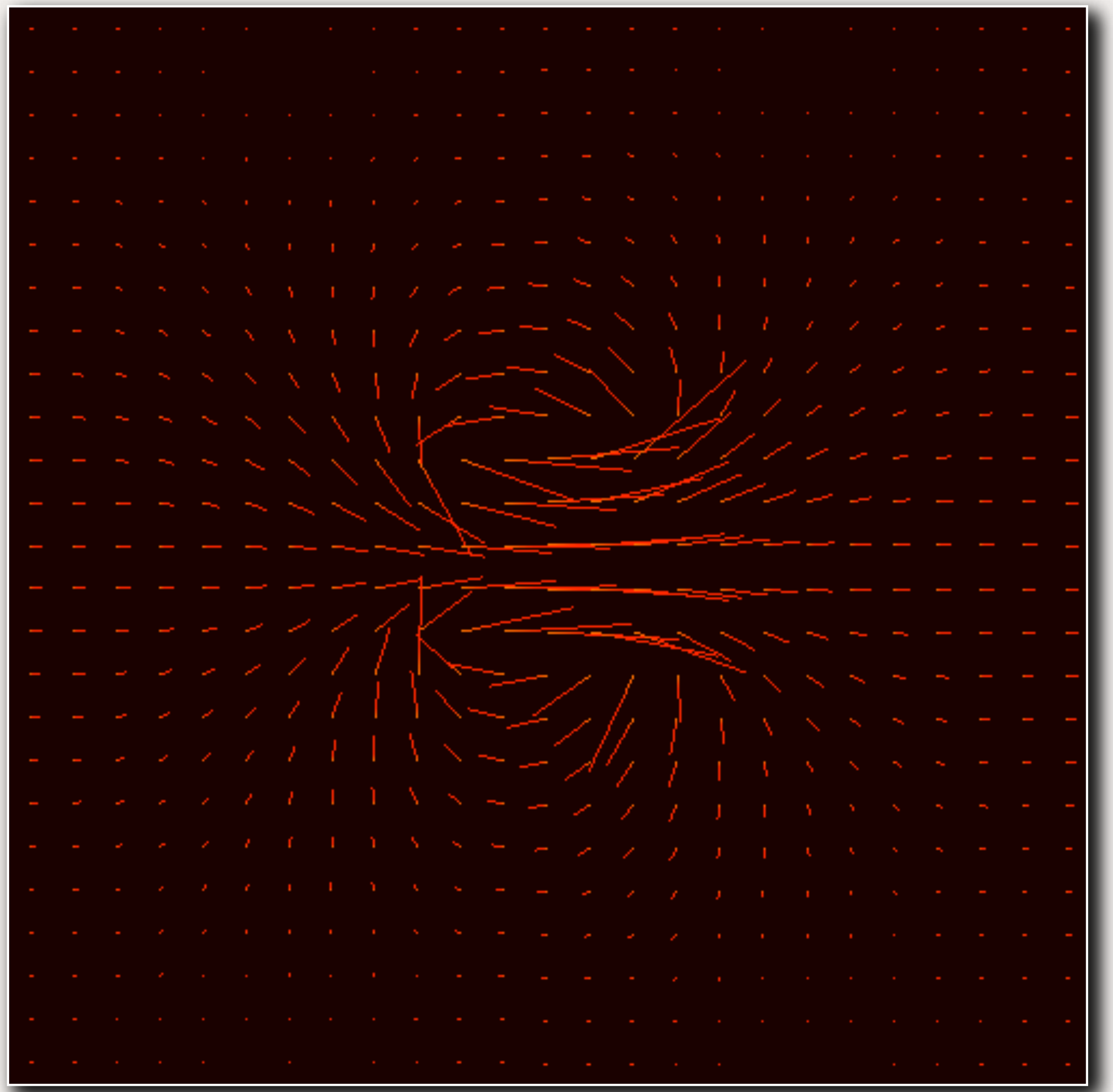
$$\boxed{\text{s.t. } \nabla \cdot \mathbf{u} = 0}$$

(divergence)

Div $\neq$ 0



Projection



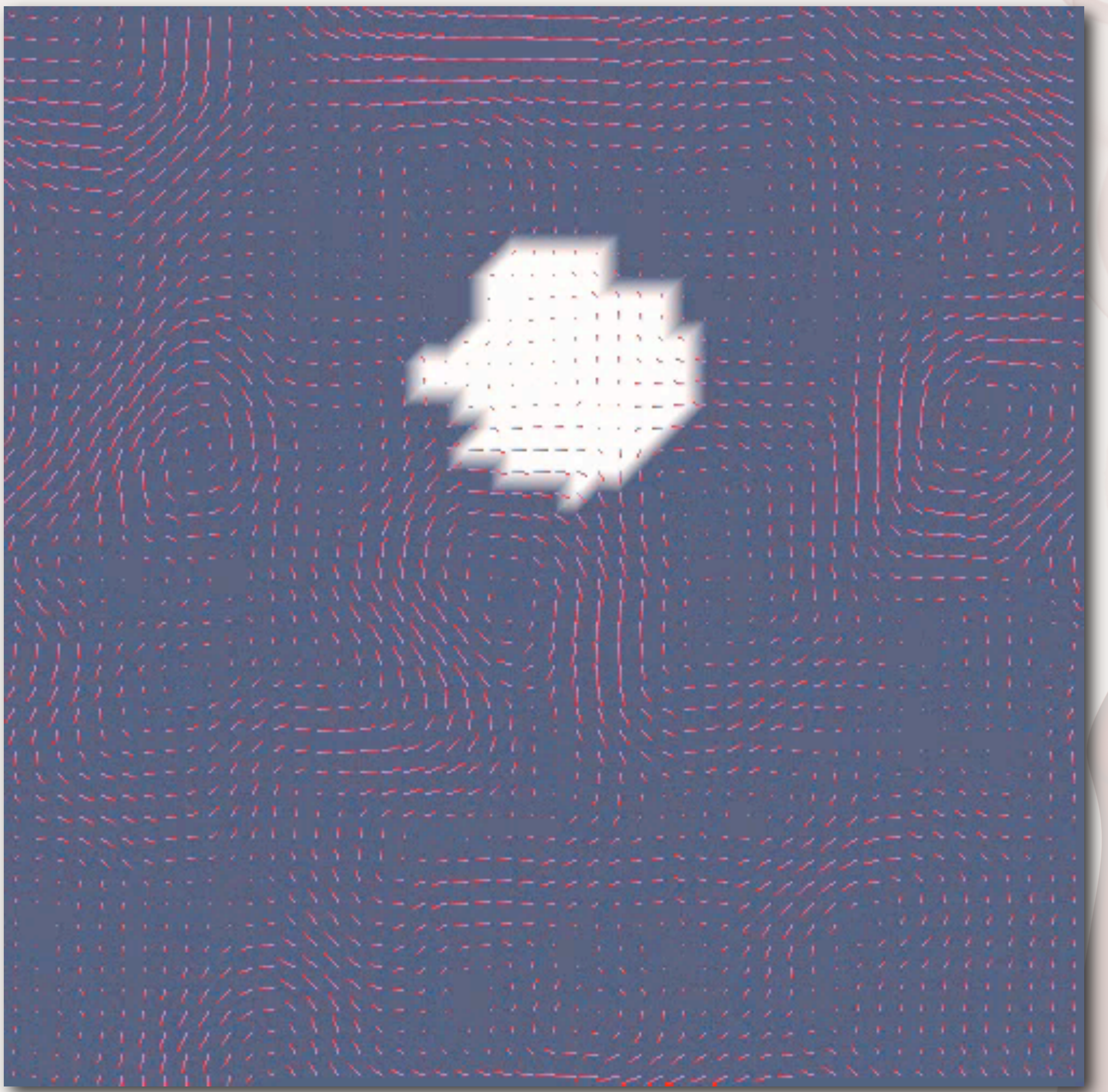
$\mathrm{Div} \neq 0$

Projection

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \boxed{\frac{1}{r}\nabla p} + s\nabla^2\mathbf{u} + \mathbf{f}$$

$$\boxed{\text{s.t.} \quad \nabla \cdot \mathbf{u} = 0}$$

Example

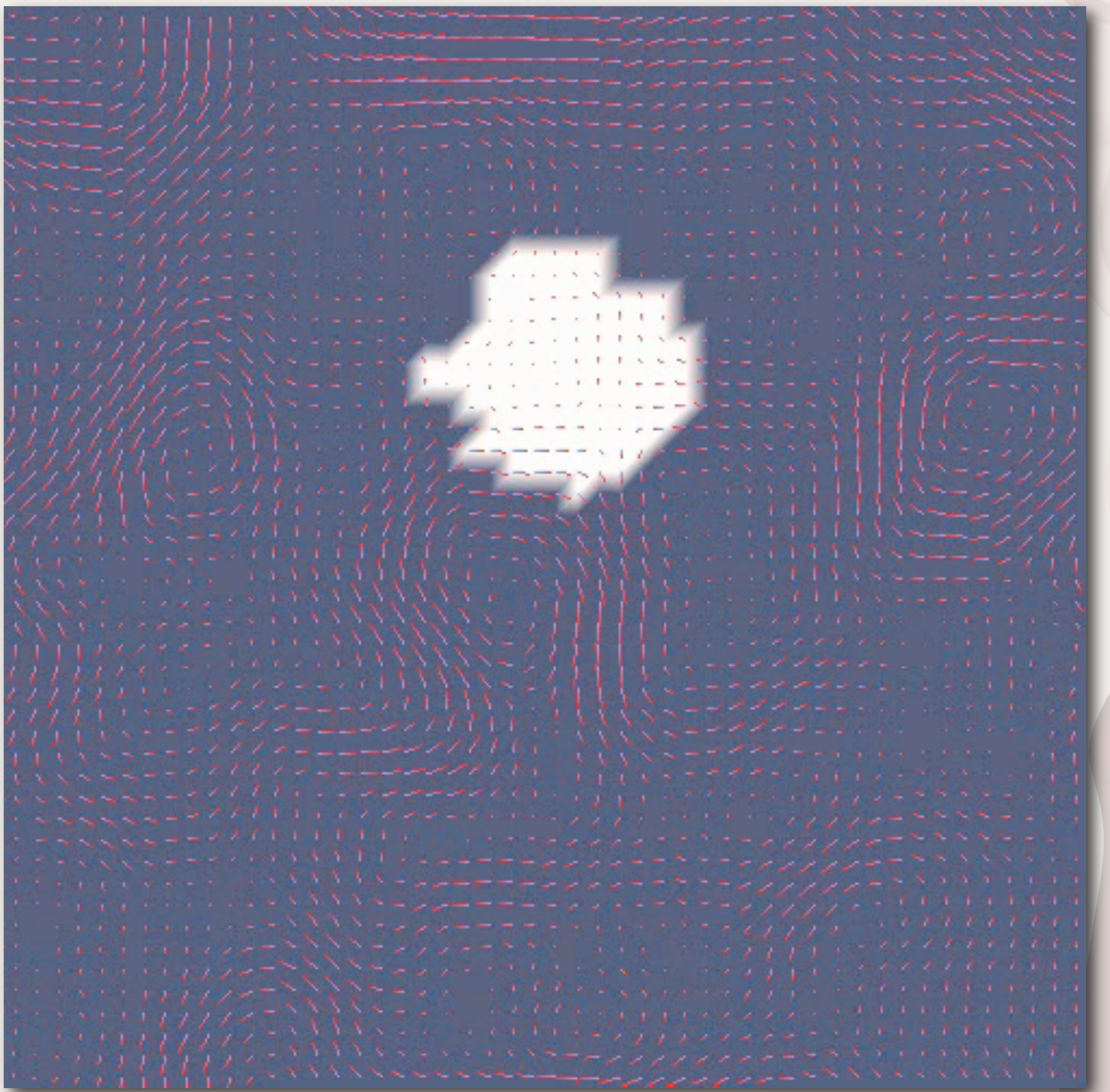


Diffusion

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + \boxed{s\nabla^2 \mathbf{u}} + \mathbf{f}$$

s.t. $\nabla \cdot \mathbf{u} = 0$

Example



External Forces

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \boxed{\mathbf{f}}$$

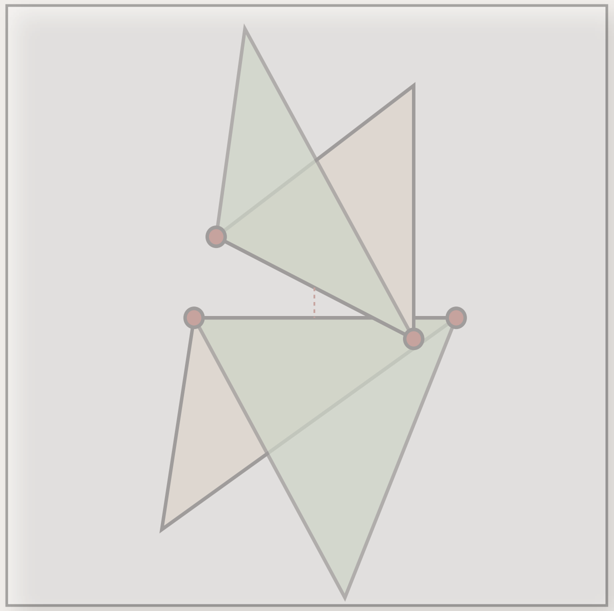
s.t. $\nabla \cdot \mathbf{u} = 0$

- **Gravity**
- **Heat**
- **Surface Tension**
- **User-Created Forces (stirring coffee)**

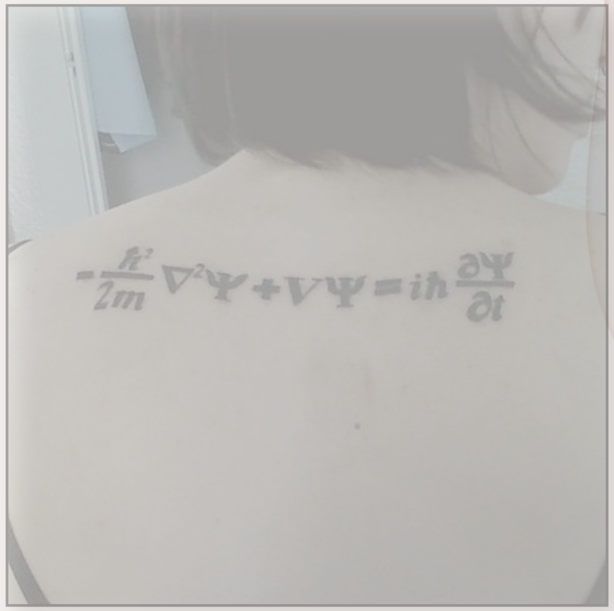
Physics Recap

- Physical quantities represented as fields.
- PDE describes the dynamics.
  - explains what we see in here…

Outline

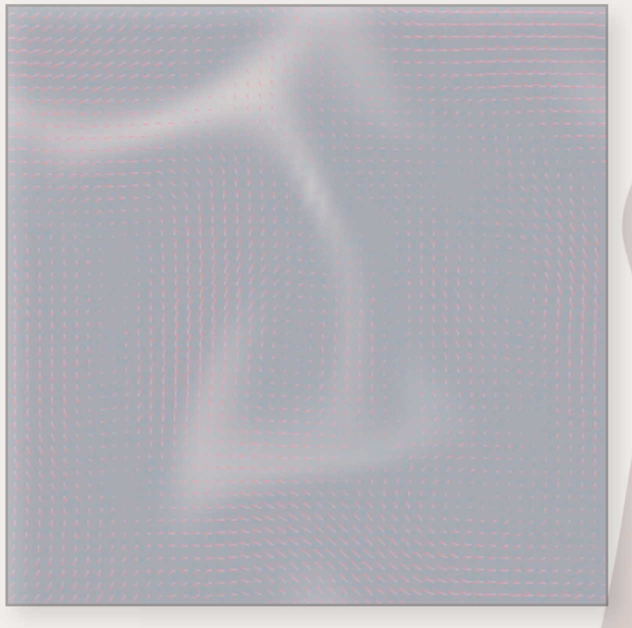


Cloth Collisions



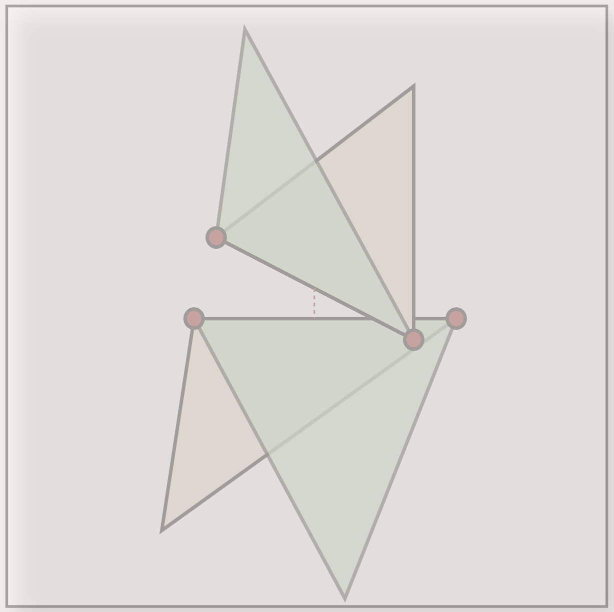
PDEs

Fluid Physics

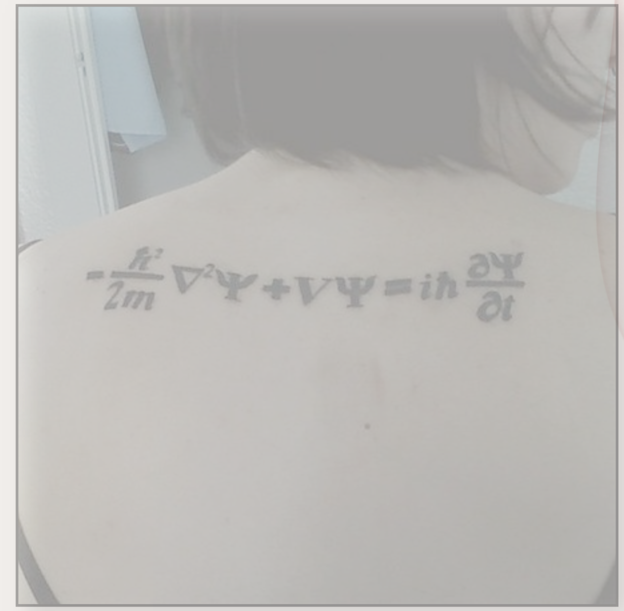


Fluid Simulation

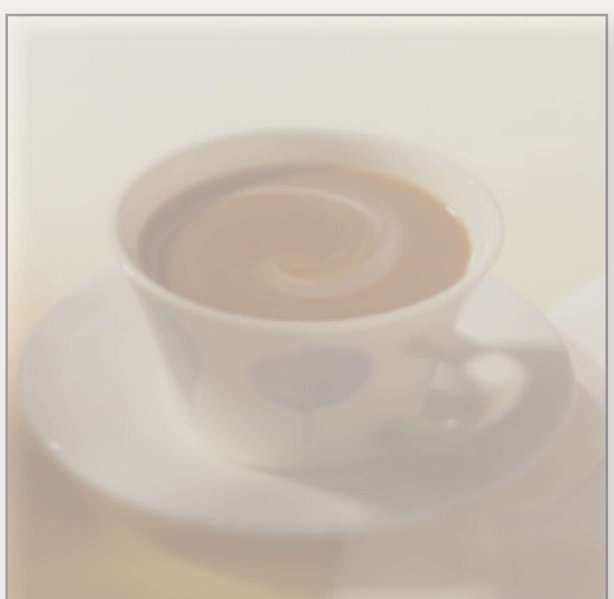
Outline



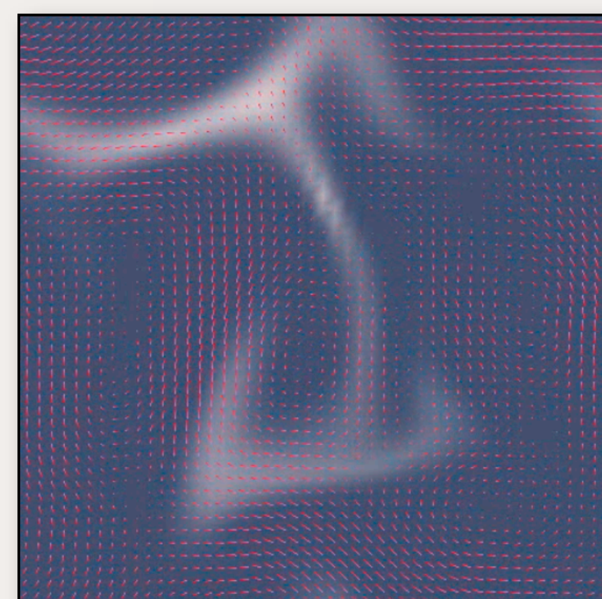
Cloth Collisions



PDEs



Fluid Physics



Fluid Simulation

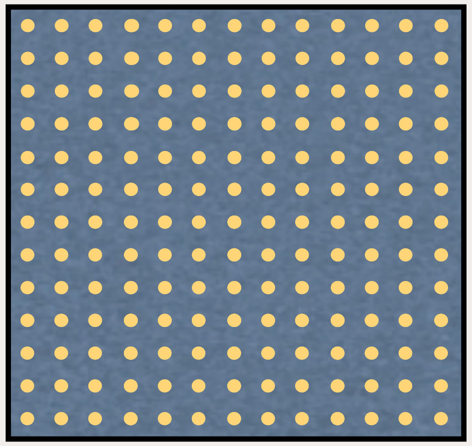
Simulation

- **Recall we're dealing with *fields*:**

$$\rho : \Omega \to [0, 1]$$
(density)

$$\mathbf{u} : \Omega \to \mathbf{R}^3$$
(velocity)

- **Grid Representation**



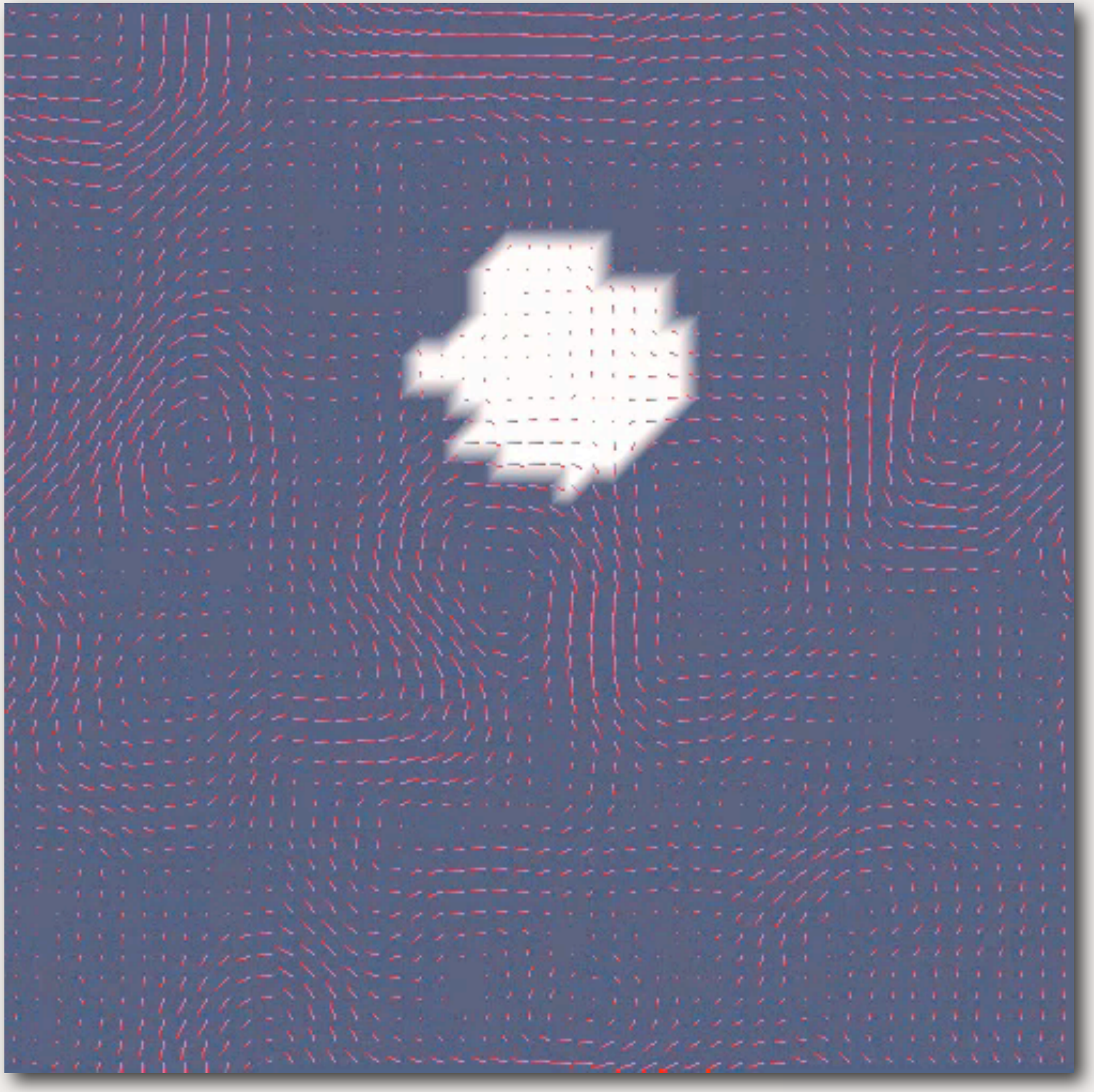
Explicit Integration

- **Very simple method to "implement" physics**

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{r}\nabla p + s\nabla^2\mathbf{u} + \mathbf{f}$$

$$x(t + \Delta t) \approx x(t) + (\Delta t) f(x(t))$$

Explicit Integration



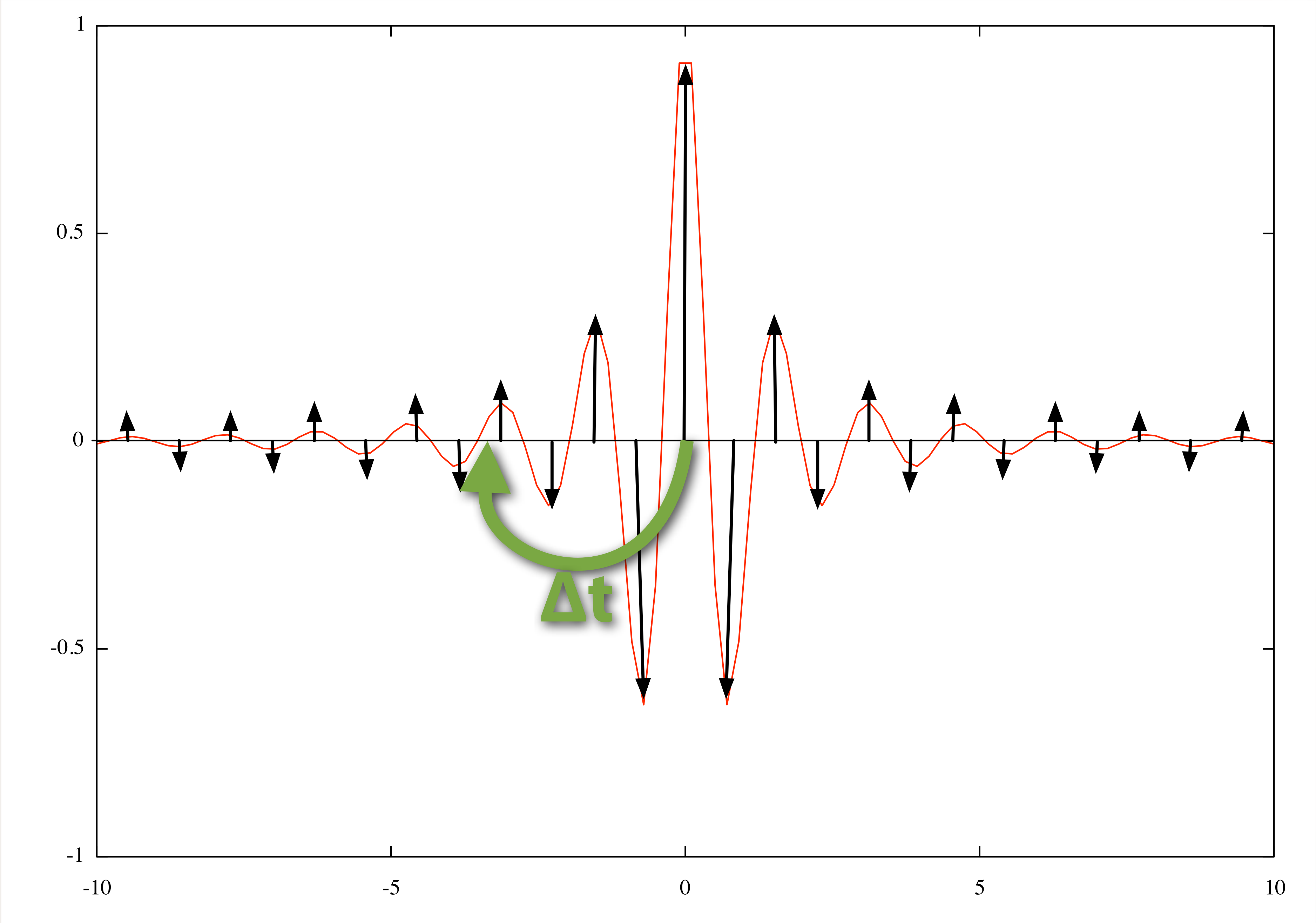
Splitting Methods

$$\frac{\partial \mathbf{u}}{\partial t} = -\boxed{(\mathbf{u} \cdot \nabla)\mathbf{u}} - \boxed{\frac{1}{r}\nabla p} + \boxed{s\nabla^2 \mathbf{u}} + \boxed{\mathbf{f}}$$

Advect → Project → Diffuse → Add Forces

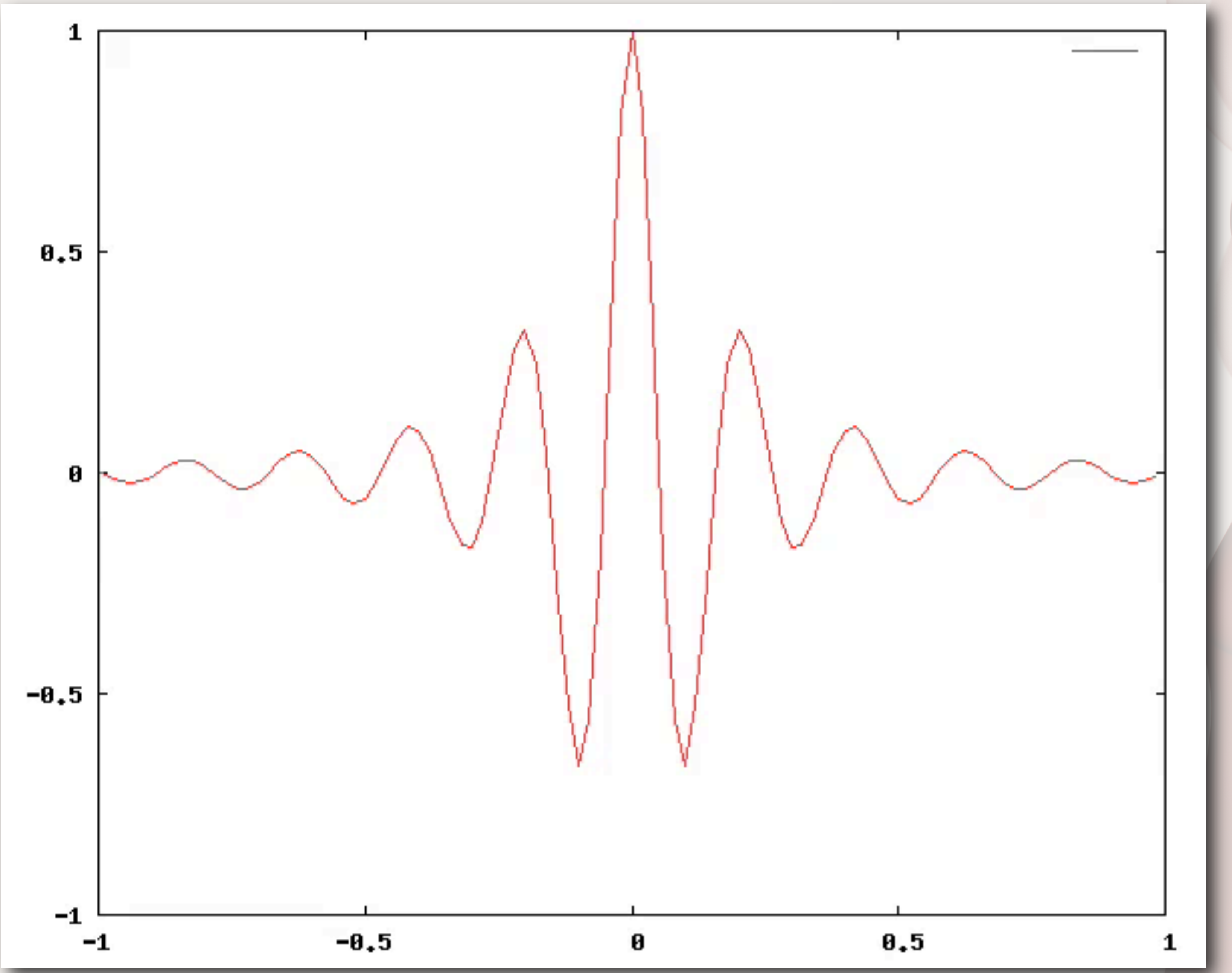
Semi-Lagrangian

$$f(x, t + \Delta t) = f(x - \Delta t, t)$$

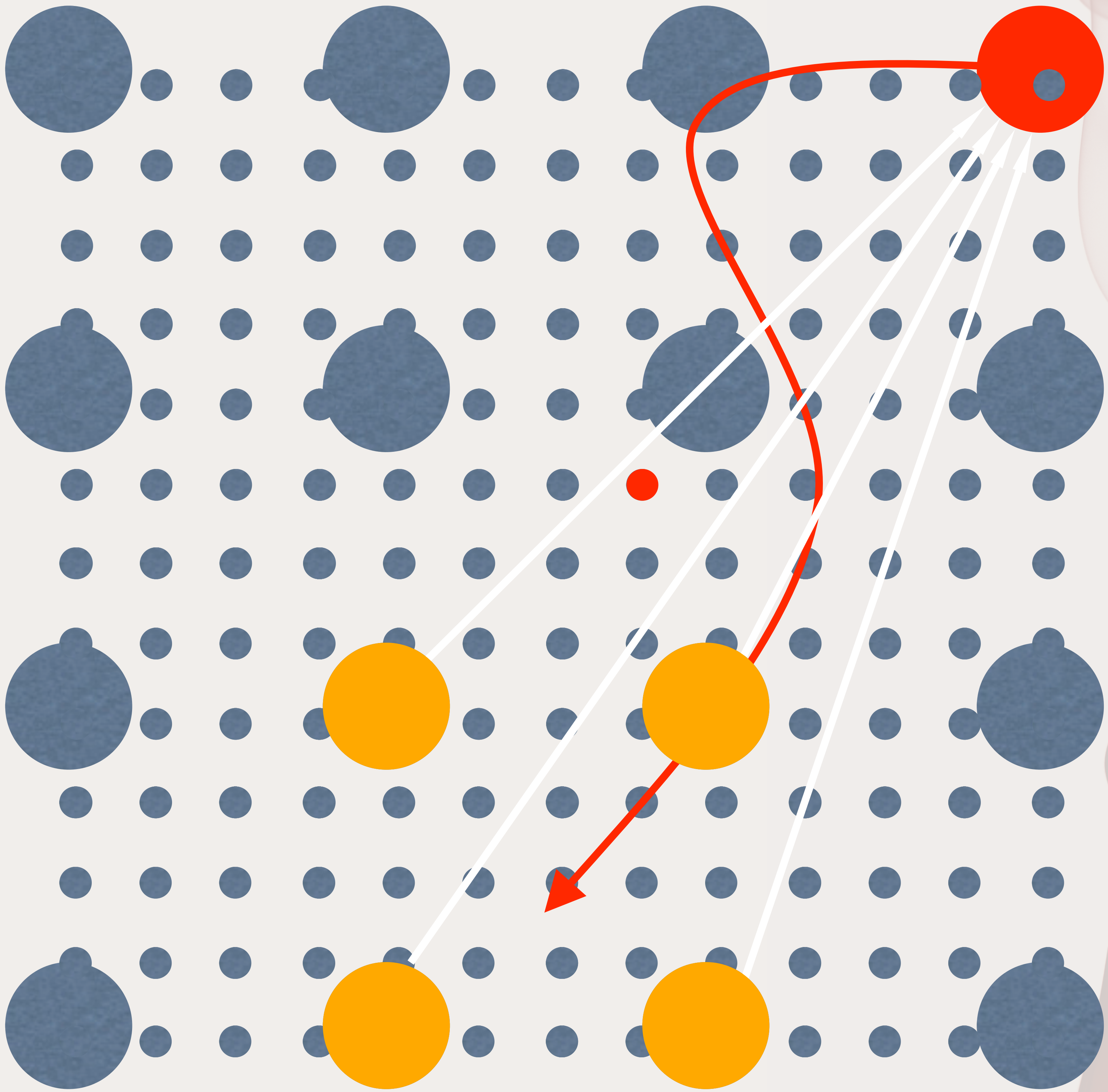


SL Advection

$$f(x, t + \Delta t) = f(x - \Delta t, t)$$

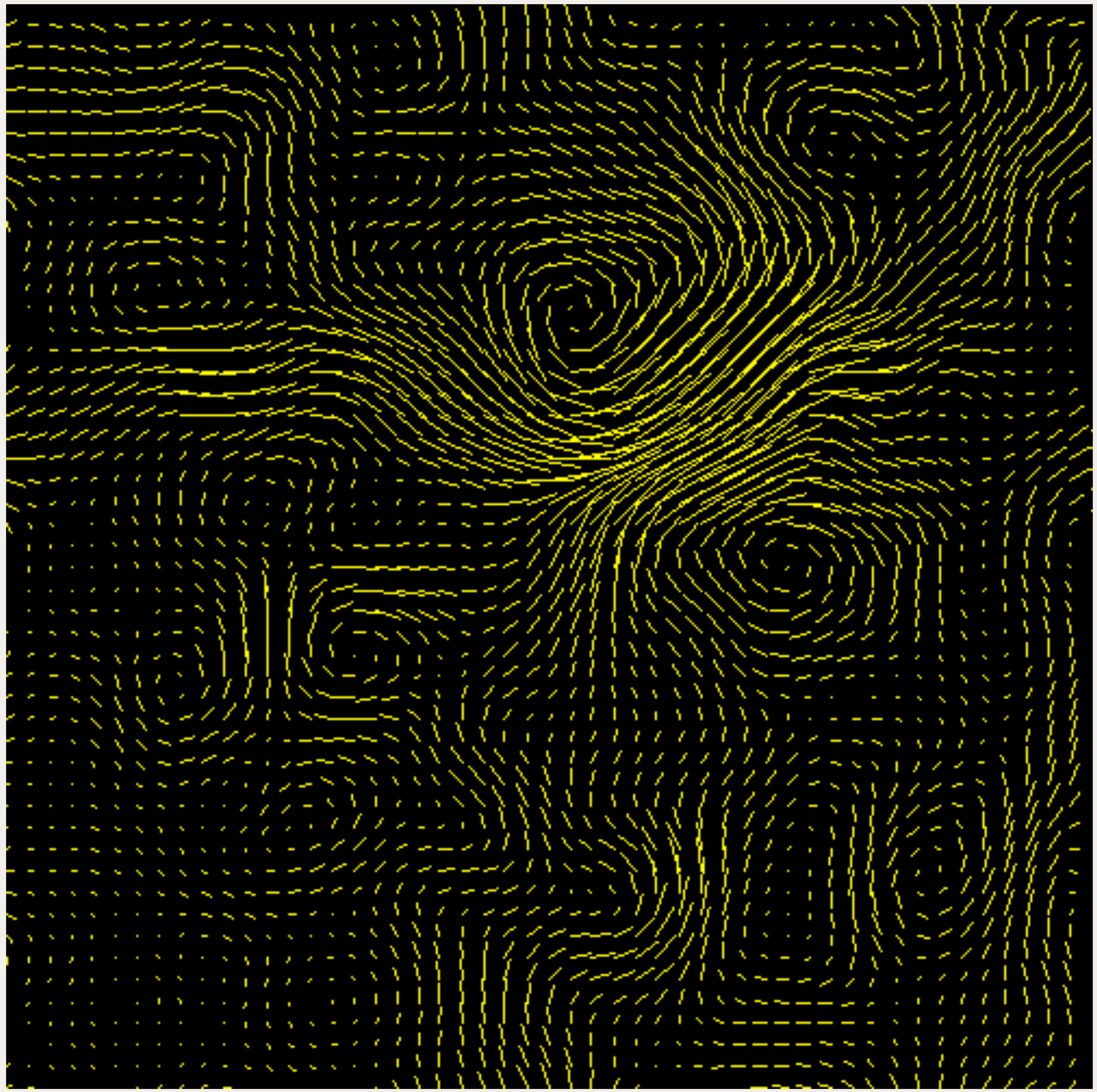


Advection



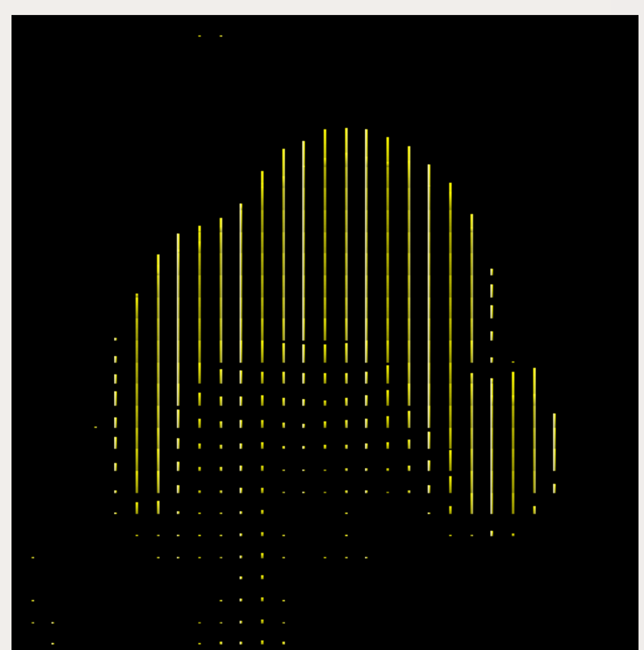
Projection

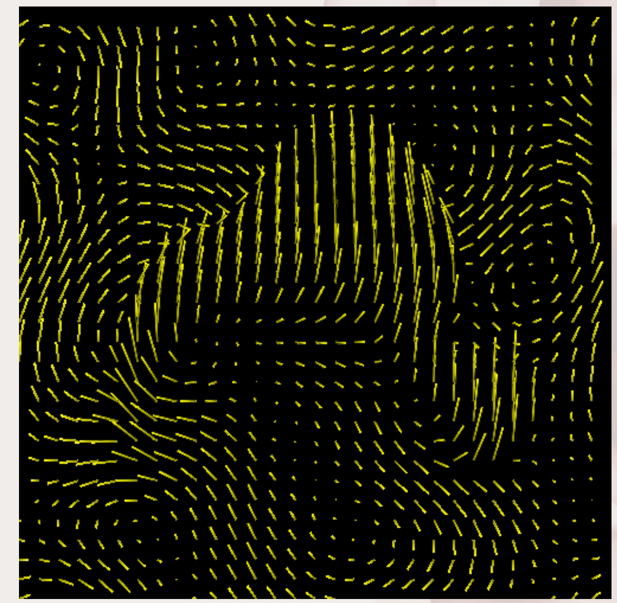
P (



)

Add Forces (e.g. heat)

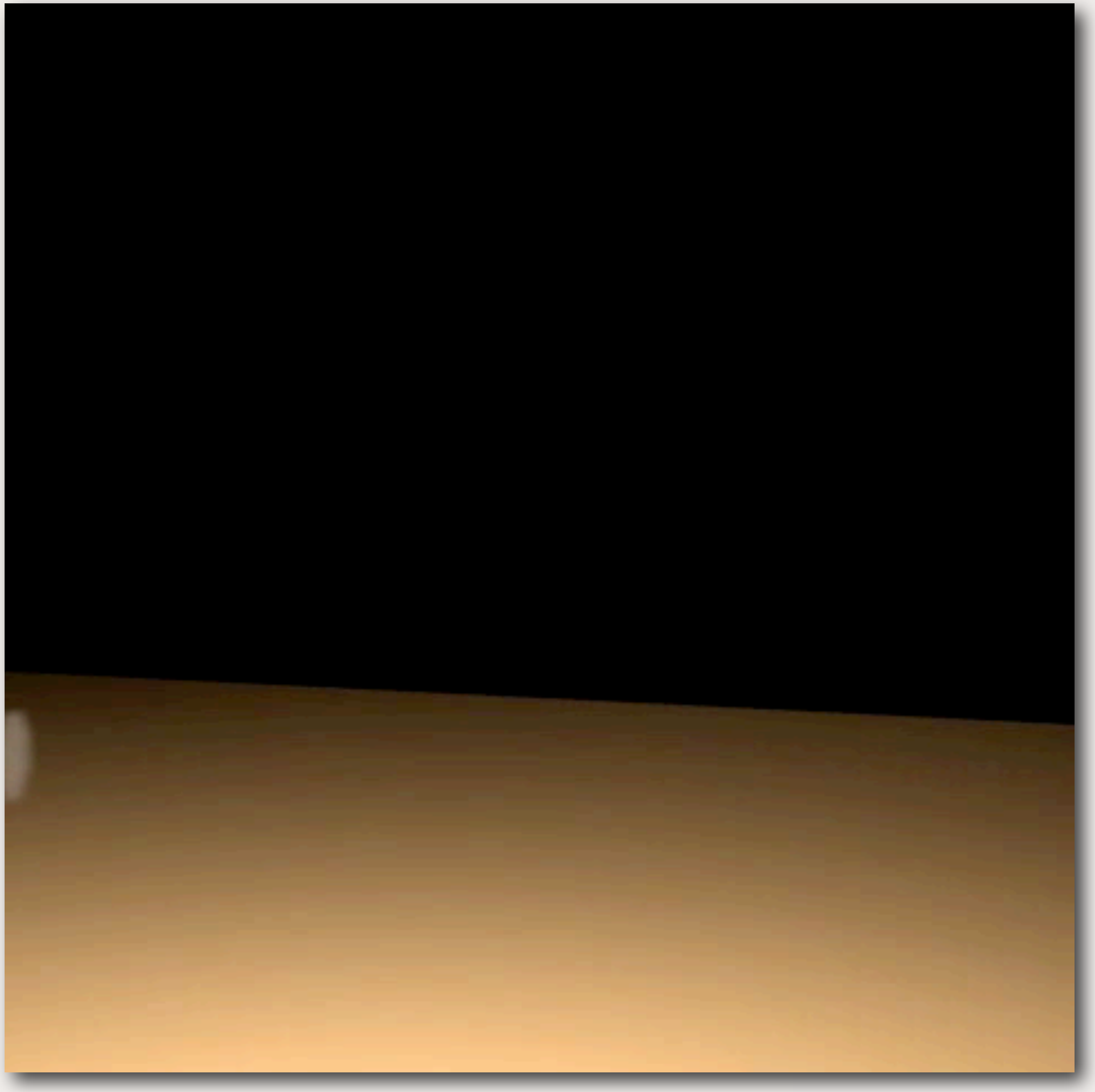




Simulation Recap

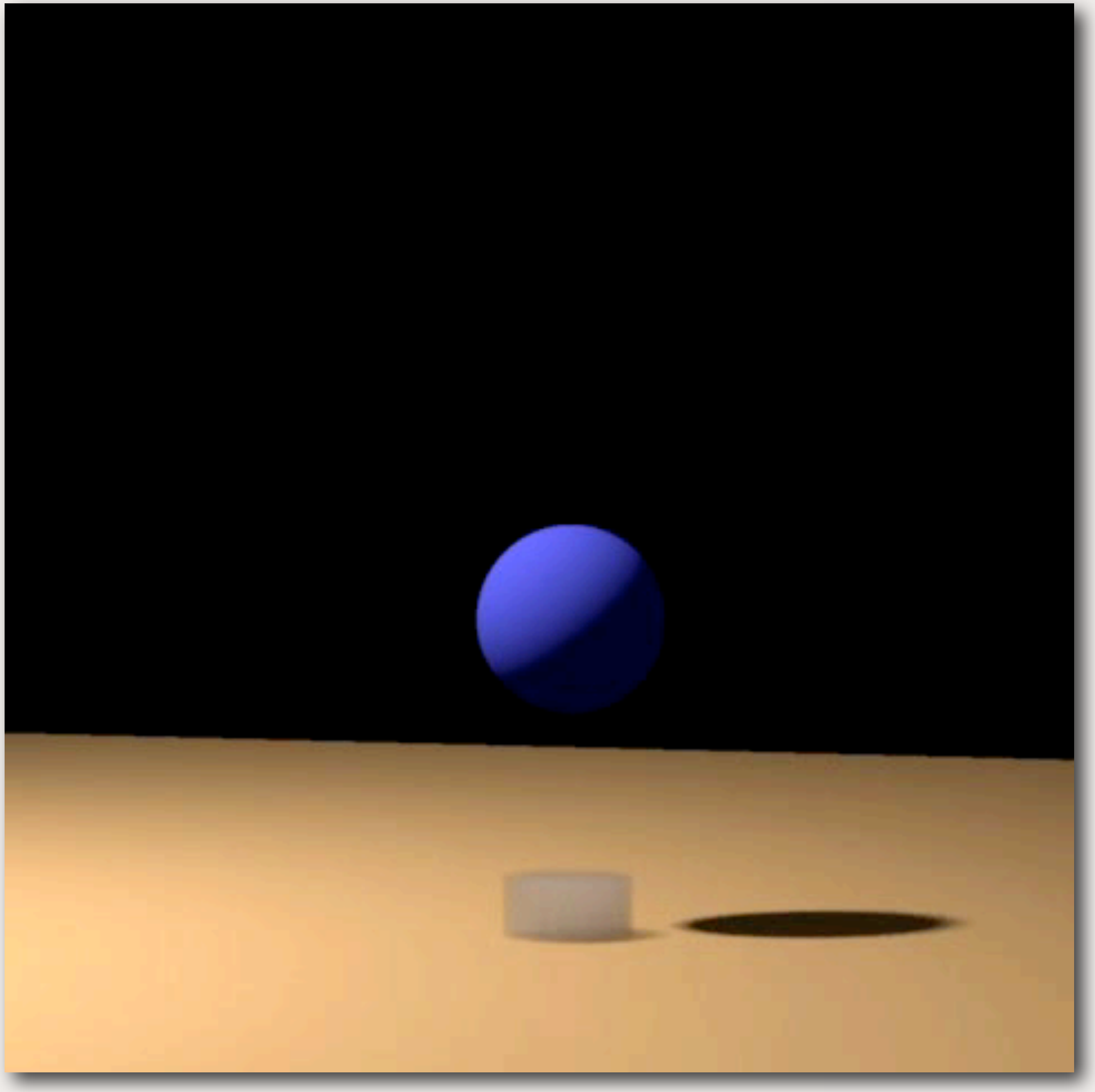
- **Decided Upon** *grid-based* **represenation.**
- **Explicit Methods will not work.**
- **Stable Fluids solves all our problems…**
  - **…maybe.**

Examples



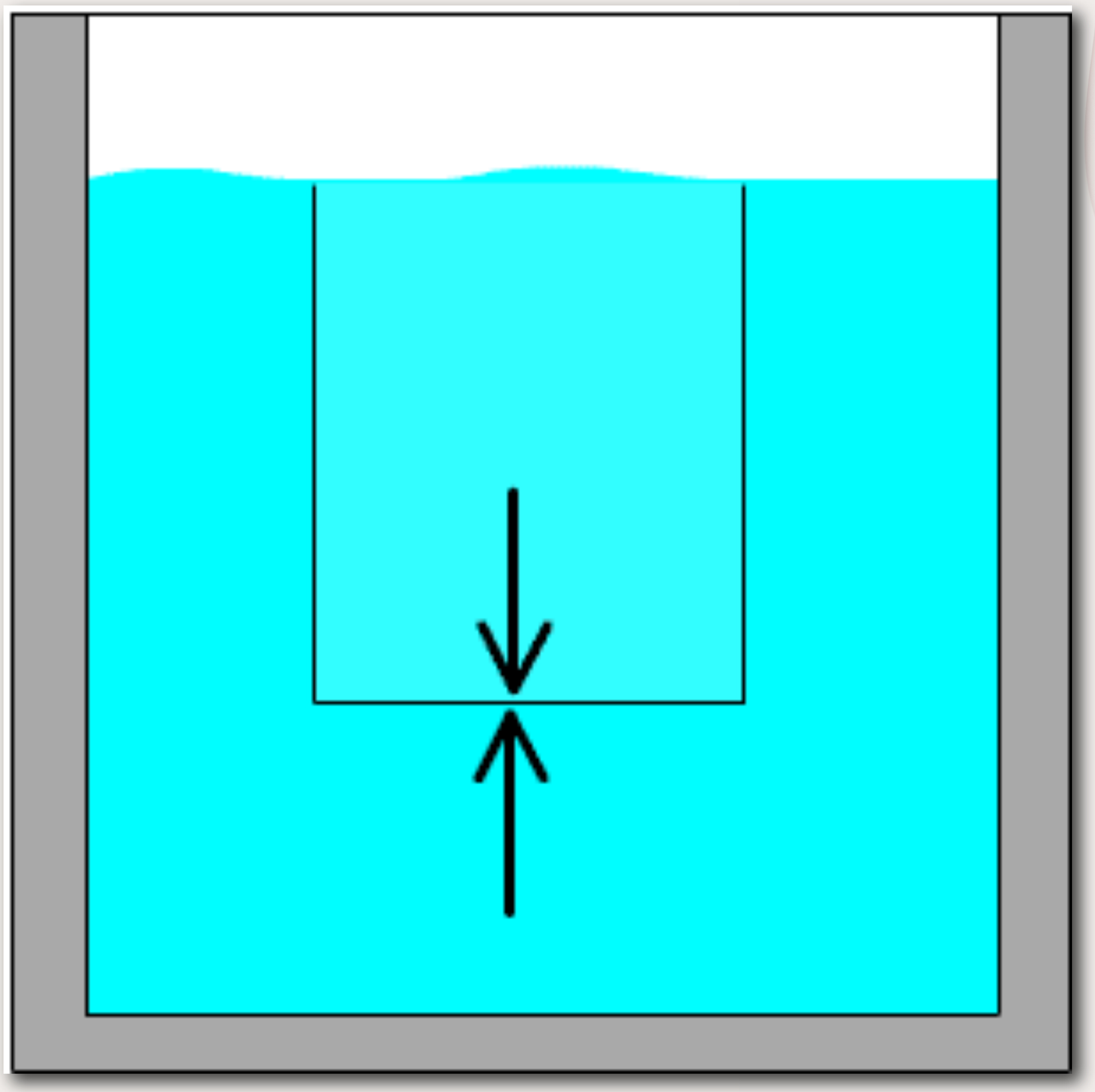
source: Fedkiw

Examples

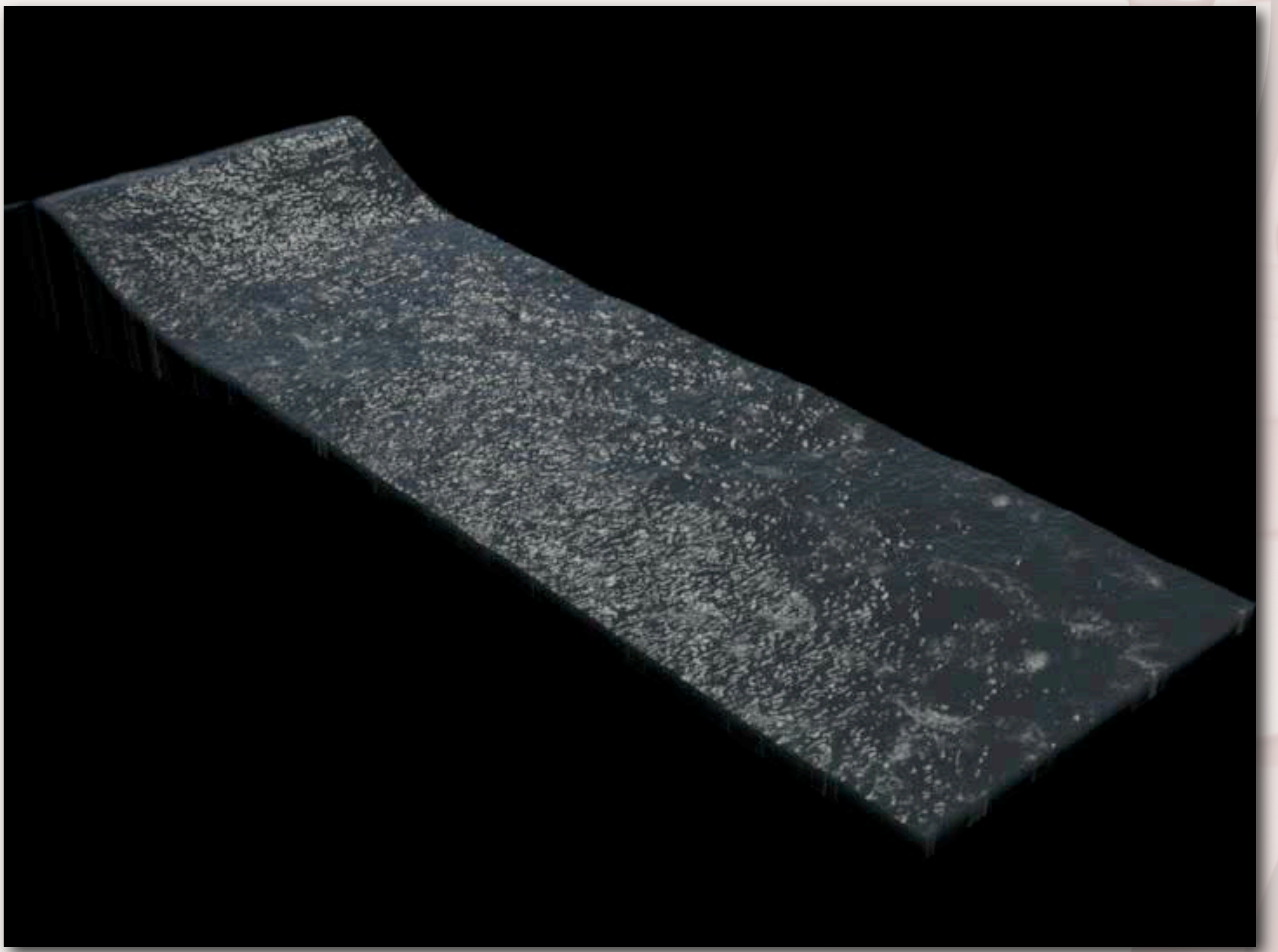


source: Fedkiw

Free Surface Fluids

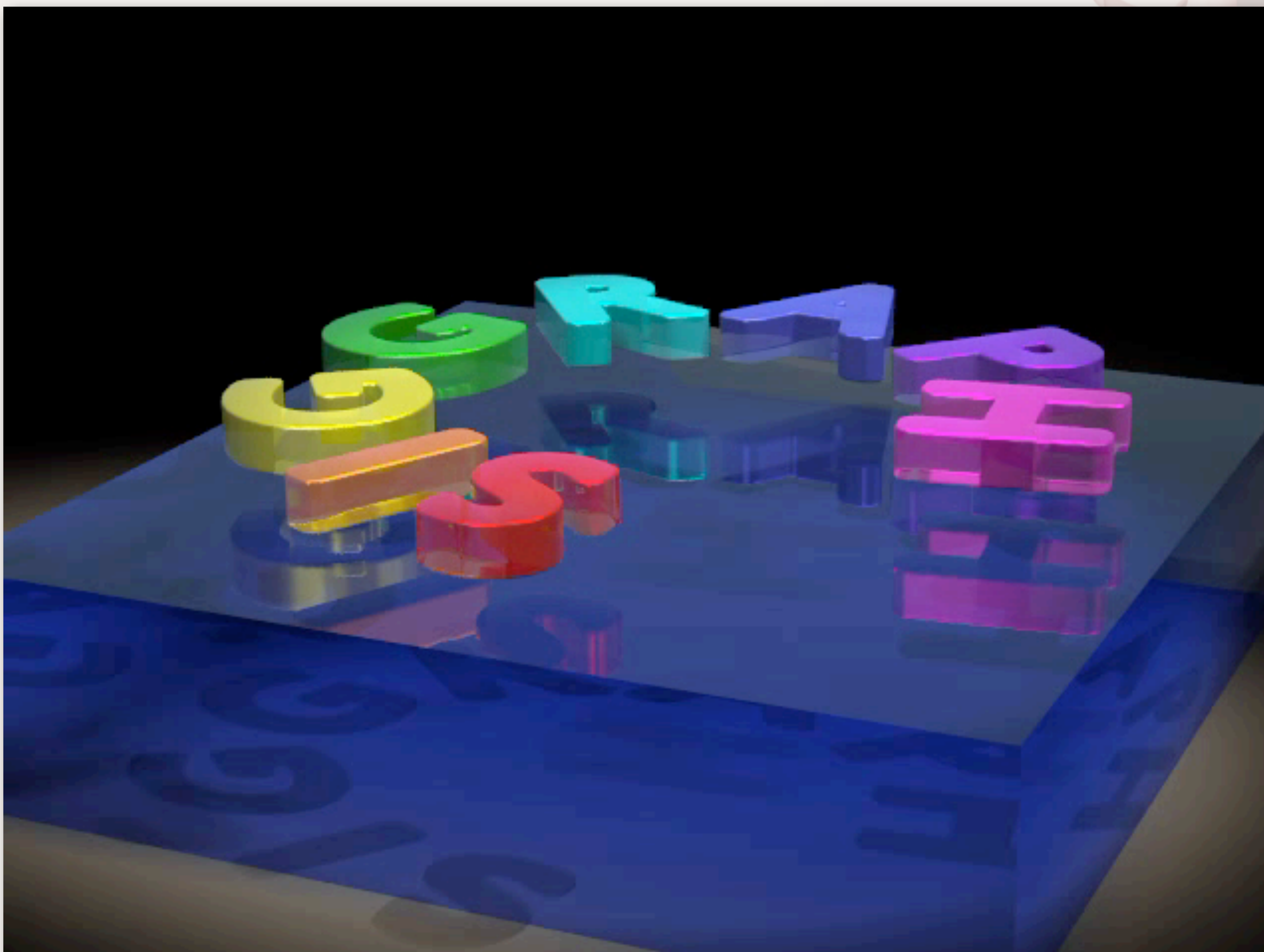


Free Surface Fluids

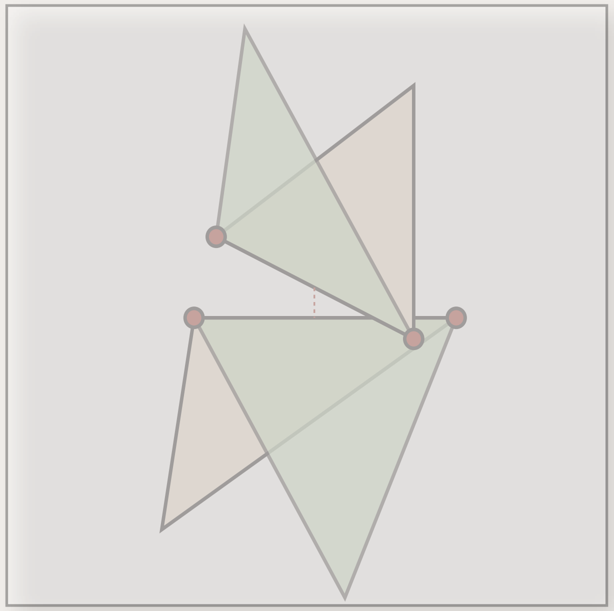


Free Surface Fluids

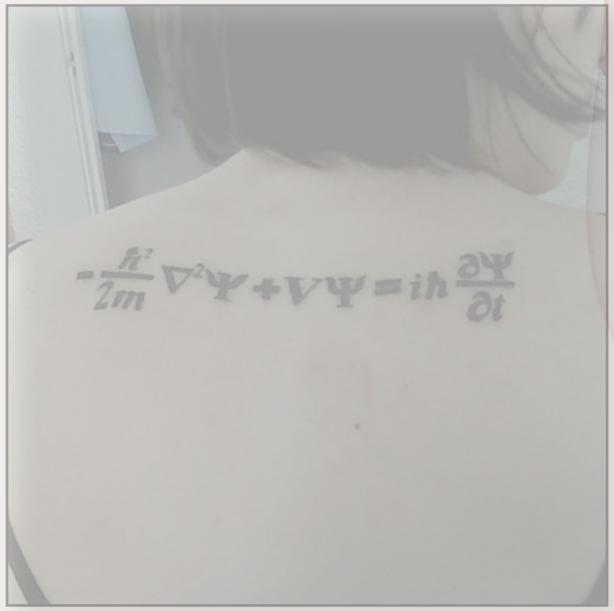
Interacting Fluids



Outline

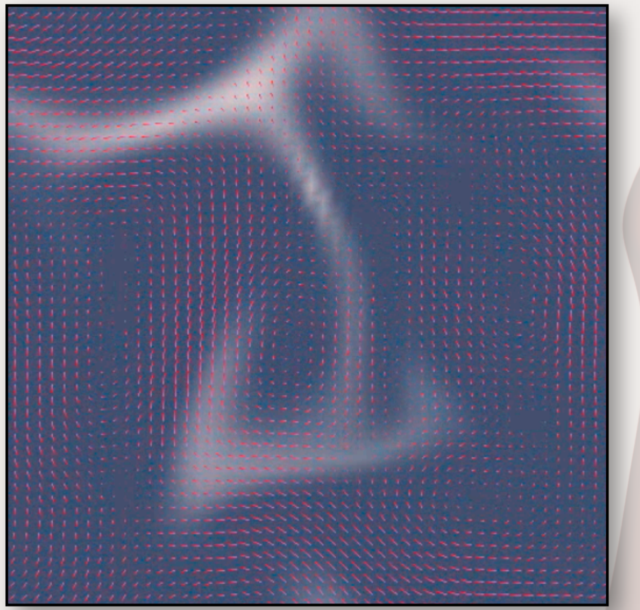


Cloth Collisions



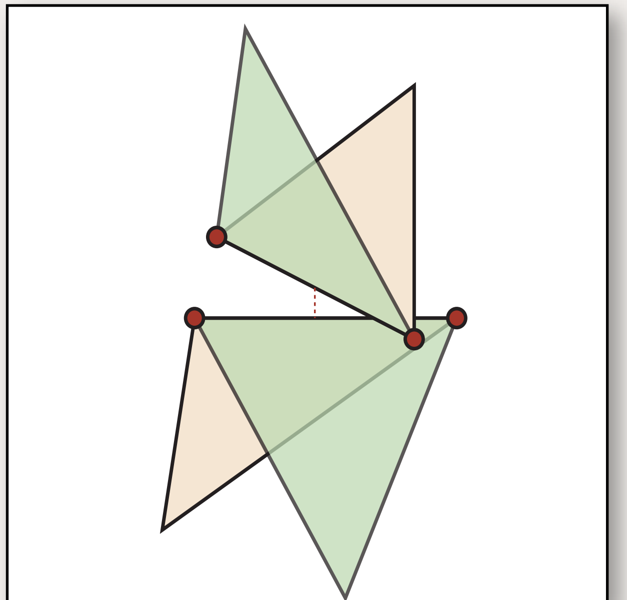
PDEs

Fluid Physics



Fluid Simulation

Outline



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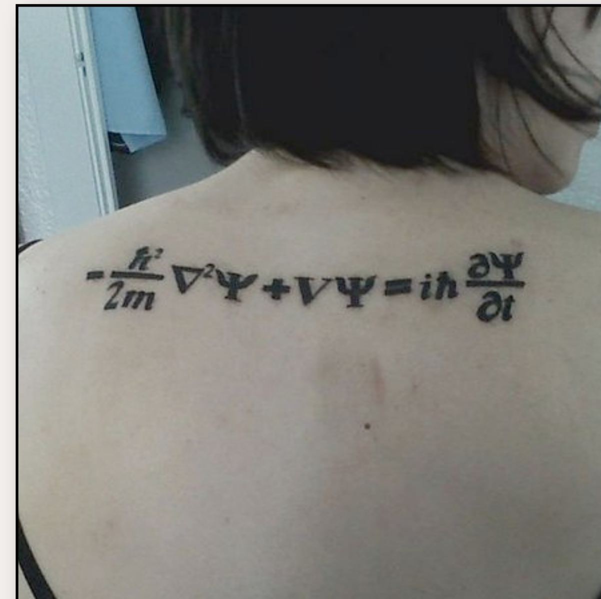
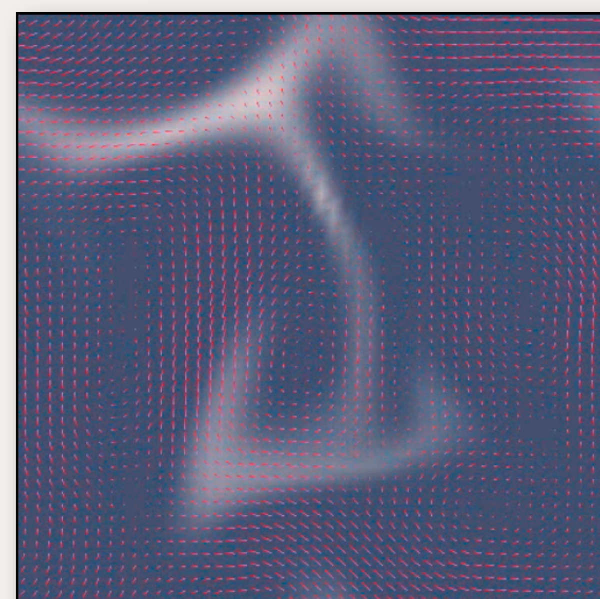


PDEs

Fluid Physics



Fluid Simulation