Ray-Sphere Intersection Slide 15

problem

$p(t) = e + td$ — ray

$f(p) = 0$ implicit surface

where do they intersect? (if at all)

intersection occur where

$f(p(t)) = 0$

$f(e + td) = 0$

sphere with center $c = (x_c, y_c, z_c)$ + radius R

is

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 - R^2 = 0$$

in vector form

$$(p - c) \cdot (p - c) - R^2 = 0 \Leftarrow \text{any such point } p \text{ is on the sphere}$$

$$(e + td - c)(e - td - c) - R^2 = 0$$

rearrange terms

$$d^2 t^2 + 2d(e - c)t + (e - c)^2 - R^2 = 0$$

quadratic equation $At^2 + Bt + c = 0$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$D = B^2 - 4AC \Leftarrow$ ~~determin~~ discriminant

$D > 0 \Rightarrow$ 2 real roots — 2 intersections with sphere

$D = 0 \Rightarrow$ 1 real root

$D < 0 \Rightarrow$ 0 real roots

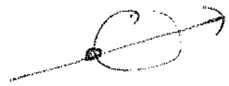
$$t = \frac{-d\cdot(e-c) \pm \sqrt{(d\cdot(e-c))^2 - d\cdot d\,((e-c)\cdot(e-c) - R^2)}}{d\cdot d}$$

(B) check value of discriminant

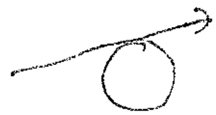
$< 0 \Rightarrow$ move on

$> 0 \Rightarrow$ 2 intersections with sphere

pick smaller value of t as closer intersection



$= 0 \Rightarrow$ grazing



Bounding volume:

only need to check discriminant

Raytracing Sphere:

find intersection point

~~compute normal~~

~~$n = 2(p-c)$~~

unit normal $\frac{(p-c)}{R}$

Ray-Triangle intersection

use barycentric coordinates
- to find intersection with parametric plane that contains △

$$x_e + t x_d = f(u,v)$$
$$y_e + t y_d = g(u,v)$$
$$z_e + t z_d = h(u,v)$$

any parametric surface

3 eq., 3 unknowns (t, u, v)

Parametric plane:

$$e + td = a + \beta(b-a) + \gamma(c-a)$$

eq. for ray — equations for plane containing our triangle.

intersection pt is in △ if + only if

$\beta > 0$

$\gamma > 0$

$\beta + \gamma < 1$

otherwise ray intersects plane outside of △

If there are no solutions then △ is degenerate or the ray is ∥ to the plane

To solve for t, β, γ expand from vector form

$$x_e + t x_d = x_a + \beta (x_b - x_a) + \gamma (x_c - x_a)$$
$$y_e + t y_d = y_a + \beta (y_b - y_a) + \gamma (y_c - y_a)$$
$$z_e + t z_d = z_a + \beta (z_b - z_a) + \gamma (z_c - z_a)$$

3X3 linear system:

$$\begin{bmatrix} x_a - x_b & x_a - x_c & x_d \\ y_a - y_b & y_a - y_c & y_d \\ z_a - z_b & z_a - z_c & z_d \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} x_a - x_c \\ y_a - y_c \\ z_a - z_c \end{bmatrix}$$

Cramer's Rule

$$\beta = \frac{\begin{vmatrix} x_a - x_e & x_a - x_c & x_d \\ y_a - y_e & y_a - y_c & y_d \\ z_a - z_e & z_a - z_c & z_d \end{vmatrix}}{|A|}$$

$$\gamma = \frac{\begin{vmatrix} x_a - x_b & x_a - x_e & x_d \\ y_a - y_b & y_a - y_e & y_d \\ z_a - z_b & z_a - z_e & z_d \end{vmatrix}}{|A|} \qquad |A| = \begin{vmatrix} x_a - x_b & x_a - x_c & x_d \\ y_a - y_b & y_a - y_c & y_d \\ z_a - z_b & z_a - z_c & z_d \end{vmatrix}$$

$$t = \frac{\begin{vmatrix} x_a - x_b & x_a - x_c & x_a - x_e \\ y_a - y_b & y_a - y_c & y_a - y_e \\ z_a - z_b & z_a - z_c & z_a - z_e \end{vmatrix}}{|A|}$$

using dummy variables for simplicity

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} j \\ k \\ l \end{bmatrix}$$

then Cramer's rule says

$$\beta = \frac{j(ei-hf) + k(gf-di) + l(dh-eg)}{m}$$

$$\gamma = \frac{i(ak-jb) + h(jc-al) + g(bl-kc)}{m}$$

$$t = \frac{f(ak-jb) + e(jc-al) + d(bl-kc)}{m}$$

where
$$m = a(ei-hf) + b(gf-di) + c(dh-eg)$$

common subterms such as
$ei-hf$ which can be computed just
once for efficiency

Algorithms for ray triangle intersection.

ray r
vector a, b, c
interval t_0, t_1 ← region we want intersections in clipping planes or closest intersection found so far

compute t
if $(t < t_0)$ or $(t > t_1)$
  return false

compute γ
if $(\gamma < 0)$ or $(\gamma > 1)$
  return false

compute β
if $(\beta < 0)$ or $(\beta > 1 - \gamma)$
  return false

intersection not within $\triangle$

start here: Slide 17

viewer sees what is in direction r

similar to last time:

$$r = d - 2(d \cdot n)n$$

again need ε - step factor not to hit original object

reflection might alter color
→ gold reflects yellow more than blue

Slide 22

Refraction

Snell's law

$$n \sin \theta = n_t \sin \phi$$

sin is more of a hassle to compute than cos

so use $\sin^2\theta + \cos^2\theta = 1$

to get

$$\cos^2\phi = 1 - \frac{n^2(1-\cos^2\theta)}{n_t^2}$$

---

need to know what t is as a 3D vector

n + b form ~~an~~ 2D orthonormal basis for the plane of refraction

$$t = \sin\phi\, b - \cos\phi\, n$$

d is known (incoming ray direction)

$$d = \sin\theta\, b - \cos\theta\, n$$

solve for b: $b = \dfrac{d + n\cos\theta}{\sin\theta}$

solve for t: ~~substituting in for b~~

$$\cancel{t = \sin\phi\left(\frac{d + n\cos\theta}{\sin\theta}\right)}$$

solve for t : substituting in for b

$$t = \sin\phi \left(\frac{d + \vec{n}\cos\theta}{\sin\theta} \right) - \cos\phi\,\vec{n}$$

sub snell's law

$$= \frac{n}{n_t}\cancel{\sin\theta} \left(\frac{d + \vec{n}\cos\theta}{\cancel{\sin\theta}} \right) - \cos\phi\, n$$

$$= \frac{n(d + \vec{n}\cos\theta)}{n_t} - \vec{n}\cos\phi$$

$$= \frac{n(d - \vec{n}(d\cdot\vec{n}))}{n_t} - \vec{n}\sqrt{1 - \frac{n^2(1 - (\vec{d}\cdot\vec{n})^2)}{n_t^2}}$$

$\sqrt{\ }$ negative? $\Rightarrow$ no refracted ray + total internal reflection