Slide 18

Cartesian vector addition:

$$\vec{a}+\vec{b} = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} + \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} x_a + x_b \\ y_a + y_b \\ z_a + z_b \end{bmatrix}$$

Cartesian dot product:

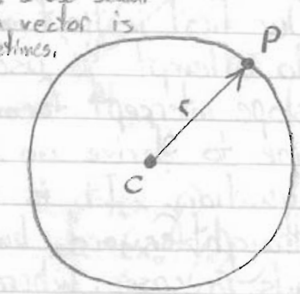
$$\vec{a} \cdot \vec{b} = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \cdot \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = x_a x_b + y_a y_b + z_a z_b$$

Cartesian cross product:

$$\vec{a} \times \vec{b} = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \times \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} y_a z_b - z_a y_b \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{bmatrix}$$

Slide 22

Implicit 2D circle: In the slides, I use scalar args, but a vector is easier sometimes.



"Suppose we want the implicit form of a circle with center $\vec{c}$ and radius r. If point $\vec{p}$ is on the circle, then the magnitude of $\vec{p}-\vec{c}$ must be r.

$$\|\vec{p}-\vec{c}\| = r$$

"Note that in order to compute a magnitude we must take a square root. However, the square of the magnitude is faster to compute because it's a simple dot product."

$$\|\vec{p}-\vec{c}\|^2 = (\vec{p}-\vec{c})\cdot(\vec{p}-\vec{c})$$

$$(\vec{p}-\vec{c})\cdot(\vec{p}-\vec{c}) = r^2$$

$$\boxed{f(\vec{p}) = (\vec{p}-\vec{c})\cdot(\vec{p}-\vec{c}) - r^2 = 0}$$

Implicit 2D line:

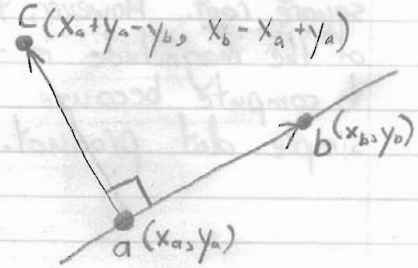
$$y = mx + b$$
$$y - mx - b = 0$$

"Our first instinct might be to attempt to use the slope-intercept form of a line to derive an implicit equation. It is very straightforward, but it fails in cases where the line is vertical."

"We want a more general form that would work for any line, like this one."

$$Ax + By + C = 0$$

"Most of the time, when we want an equation for a line in graphics, we know two points $\vec{a}$ and $\vec{b}$. How do we find an appropriate A, B, and C coefficients given two points?"



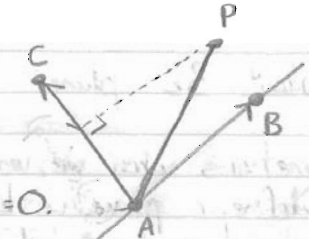
"The first step is to define a third point such that $\|c-a\| = \|b-a\|$ and $(c-a)$ is perpendicular to $(b-a)$"

$$b - a = \begin{bmatrix} x_b - x_a \\ y_b - y_a \end{bmatrix}$$

$$c - a = \begin{bmatrix} y_a - y_b \\ x_b - x_a \end{bmatrix}$$

$$c = \begin{bmatrix} y_a - y_b + x_a \\ x_b - x_a + y_a \end{bmatrix}$$

"Now we know that if a point p is on the line, $(\vec{p} - \vec{a})$ is perpendicular to $(\vec{c} - \vec{a})$, so $(\vec{p} - \vec{a}) \cdot (\vec{c} - \vec{a}) = 0$. Otherwise, the dot product will be non-zero."



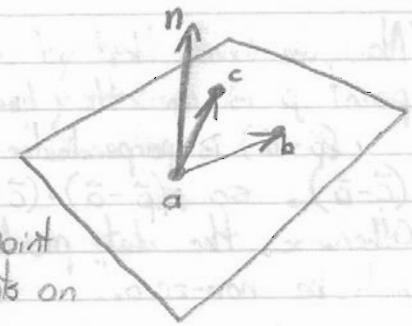
"In fact, this dot product is a signed, scaled distance of p from the line, which will be useful later."

$$p = (x, y)$$
$$a = (x_a, y_a)$$
$$b = (x_b, y_b)$$
$$c - a = (y_a - y_b,\ x_b - x_a)$$
$$p - a = (x - x_a,\ y - y_a)$$

$$(\vec{p} - \vec{a}) \cdot (\vec{c} - \vec{a})$$
$$= (x - x_a)(y_a - y_b) + (y - y_a)(x_b - x_a)$$
$$= x(y_a - y_b) + y(x_b - x_a) - \cancel{x_a y_a} + x_a y_b - y_a x_b + \cancel{x_a y_a}$$
$$= x(y_a - y_b) + y(x_b - x_a) + x_a y_b - y_a x_b$$

$$\boxed{f(x, y) = x(y_a - y_b) + y(x_b - x_a) + x_a y_b - y_b x_a = 0}$$

Implicit 3D plane:



"Sometimes when we want to define a plane, we know a normal vector $\vec{n}$ and a point $\vec{a}$. If we know three points on the plane instead, we can obtain a normal using a cross product."

$$\vec{n} = (\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})$$
$$\vec{n} \cdot (\vec{p}-\vec{a}) = 0$$

"Then, we know for every point $\vec{p}$ on the plane, $\vec{n} \cdot (\vec{p}-\vec{a}) = 0$, and it's non-zero otherwise, so

$$f(\vec{p}) = \vec{n} \cdot (\vec{p}-\vec{a}) = 0$$

is an implicit equation for a plane.

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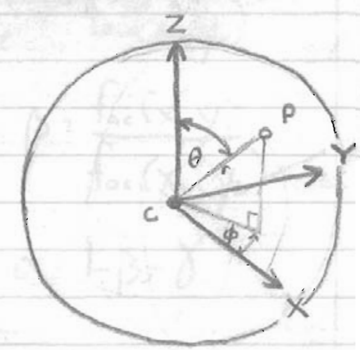
Parametric 3D line:

"If we know two points $\vec{a}$ and $\vec{b}$ on the line, we can easily obtain a parametric form by assuming $f(0)=a$ and $f(1)=b$."

$f(0)=a$
$f(1)=b$

$$f(t) = a + t(b-a)$$

Parametric sphere:



Suppose a sphere w/ center c and radius r. Let ϕ be the angle corresponding with longitude $(-180°, 180°]$ and θ be the angle corresponding with lattitude.

$$x = x_c + r\cos\phi\sin\theta$$
$$y = y_c + r\sin\phi\sin\theta$$
$$z = z_c + r\cos\theta$$

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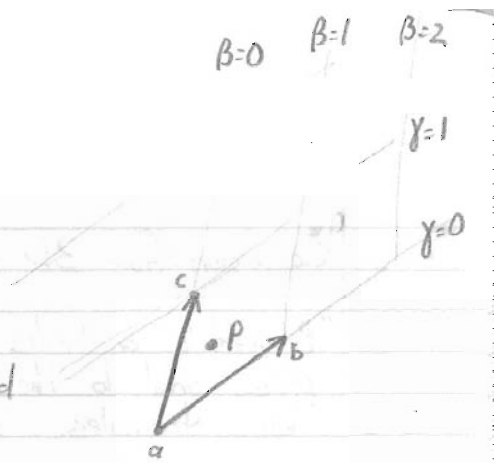


Conversion from 2D Cartesian

"Note that a barycentric coord system can have gridlines like a Cartesian system."

"The b.c. components are just signed, weighted distances from the axes."

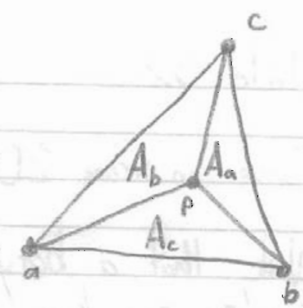
"We know that the function of the implicit form for a line gives a signed, weighted distance to the line, but the weights are probably not such that $f_{ab}(x_c, y_c) = 1$, for example, so we divide."

$$\gamma = \frac{f_{ab}(x, y)}{f_{ab}(x_c, y_c)}$$

$$\beta = \frac{f_{ac}(x, y)}{f_{ac}(x_b, y_b)}$$

$$\alpha = 1 - \beta - \gamma$$

Conversion from 3D Cartesian



"Barycentric coordinates are also proportional to the signed area of these triangles."

$$\text{area} = \frac{1}{2} \|(b-a) \times (c-a)\|$$

$$\alpha = \frac{n \cdot n_a}{\|n\|^2} \qquad \beta = \frac{n \cdot n_b}{\|n\|^2} \qquad \gamma = \frac{n \cdot n_c}{\|n\|^2}$$

$$n_a = (c-b) \times (p-b)$$
$$n_b = (a-c) \times (p-c)$$
$$n_c = (b-a) \times (p-a)$$