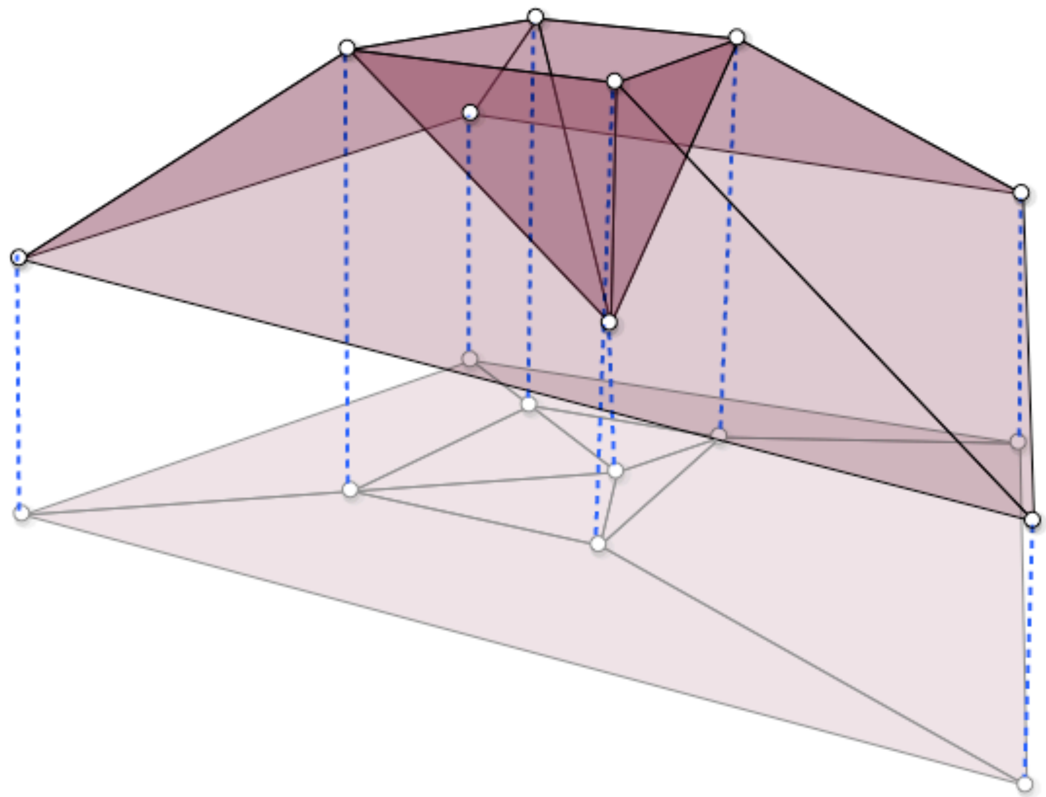
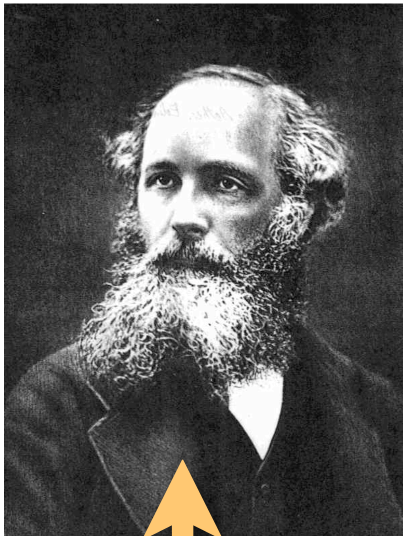
Planar Graphs in $2^1/_2$ Dimensions

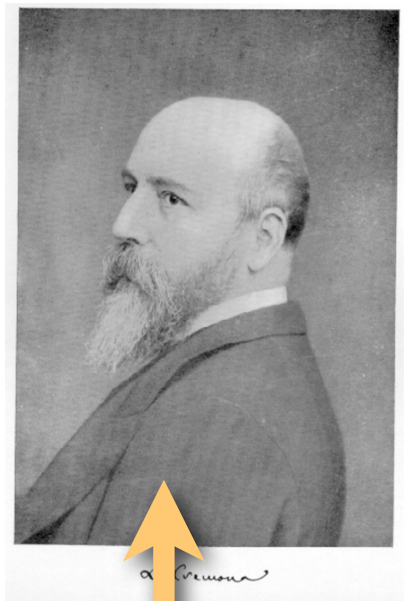
Don Sheehy

$2^1/_2$ Dimensions



Cast of Characters





Ernst Steinitz

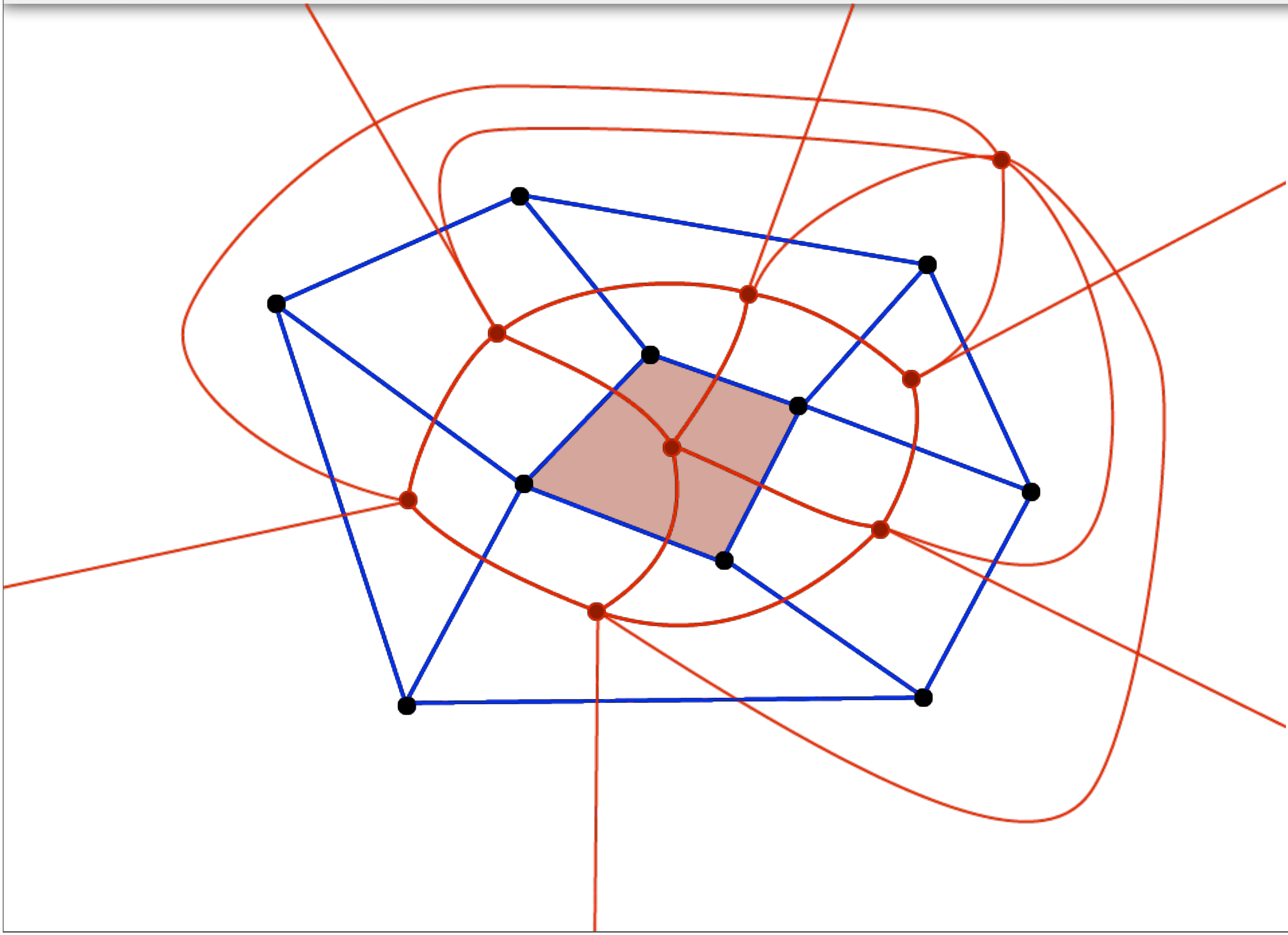
James Clerk Maxwell

Luigi Cremona

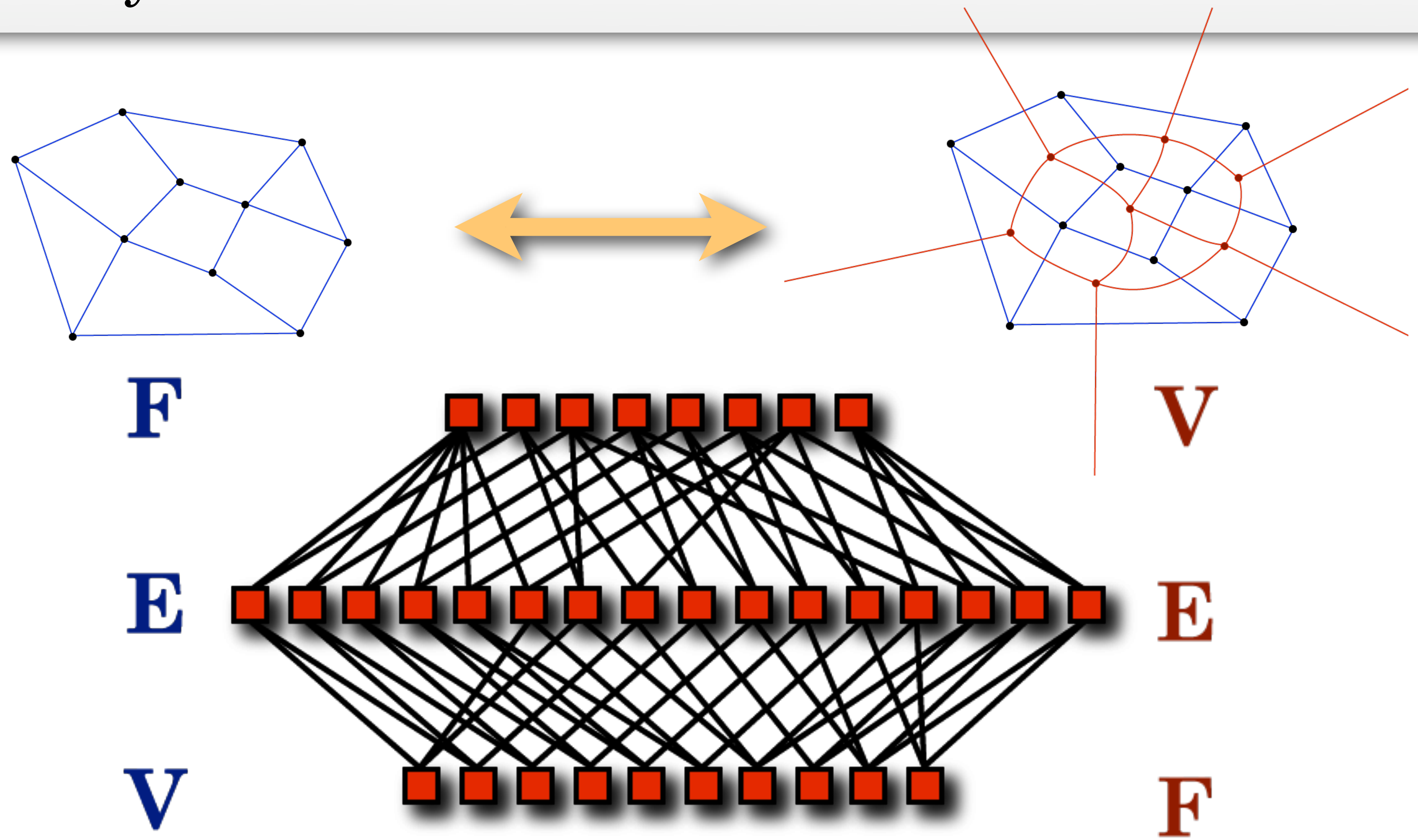
W. T. Tutte

Planar Graphs

Planar Graphs

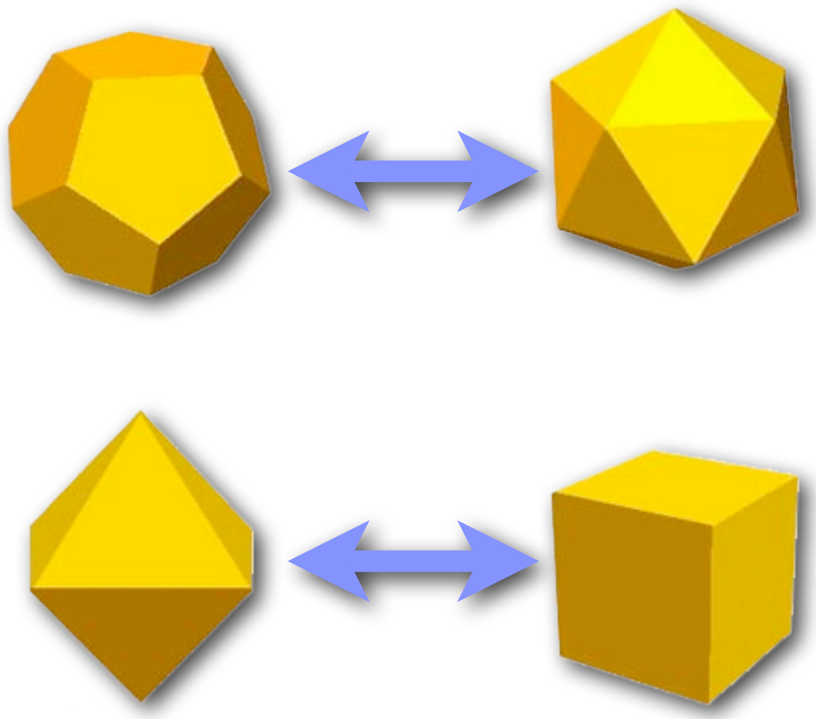


Duality



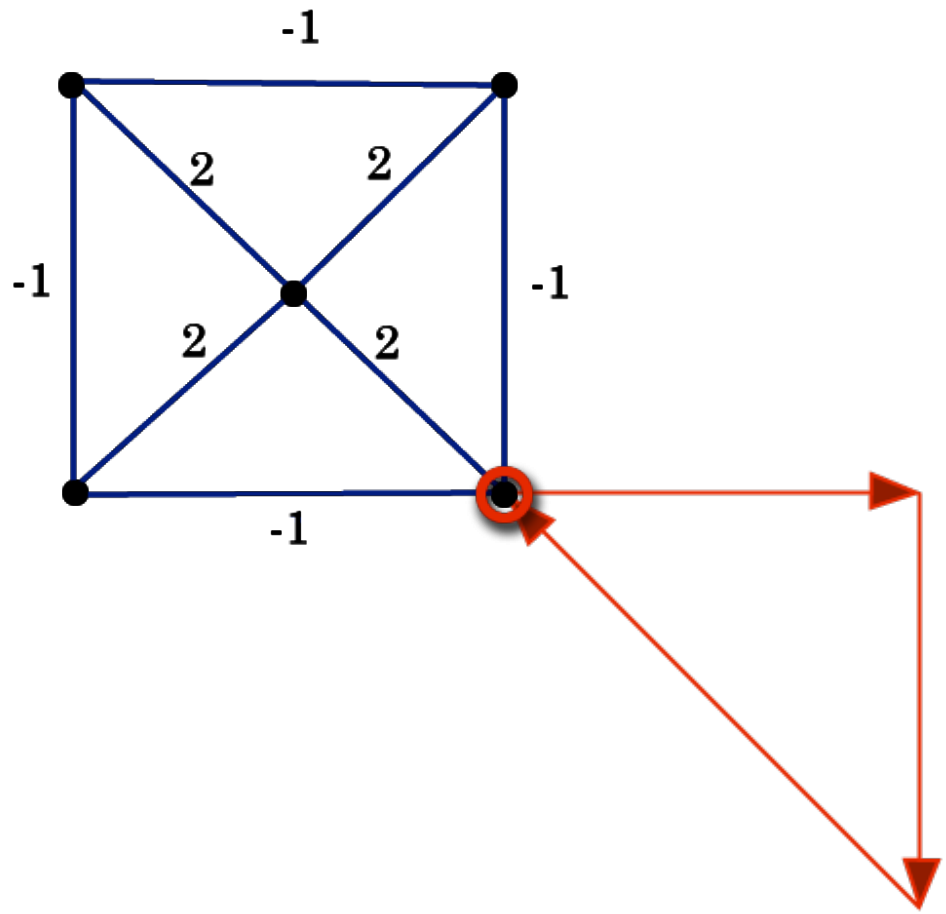
Polar Polytopes

$$A^\circ = \{x \in \mathbb{R}^d \mid a \cdot x \leq 1, \forall a \in A\}$$



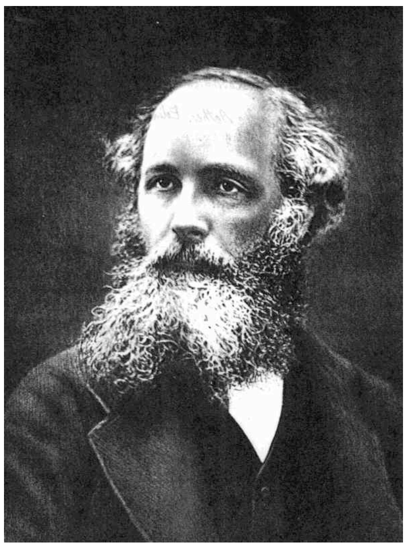
The Maxwell-Cremona Correspondence

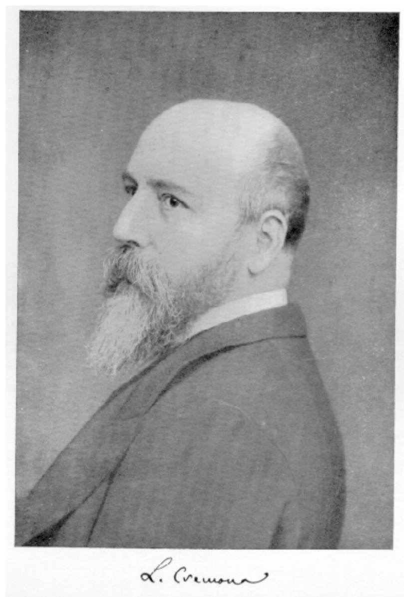
Equilibrium Stresses



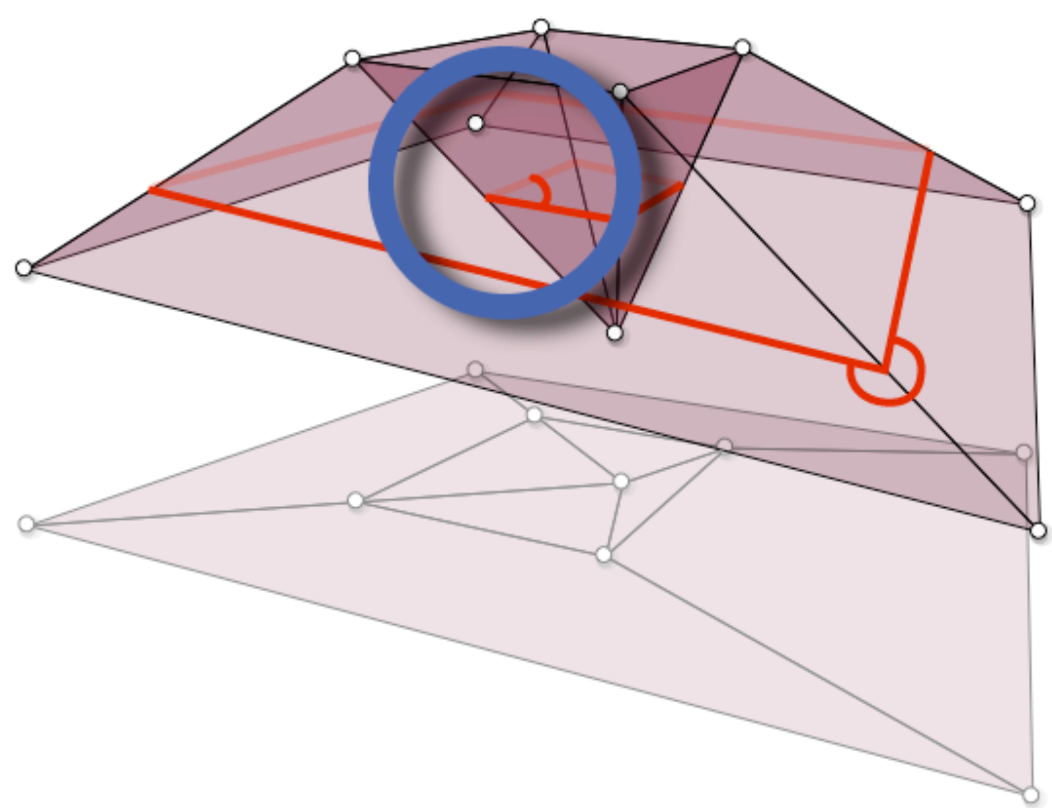
The Maxwell-Cremona Correspondence

There is a 1-1 correspondence between "proper" liftings and equilibrium stresses of a planar straight line graph.

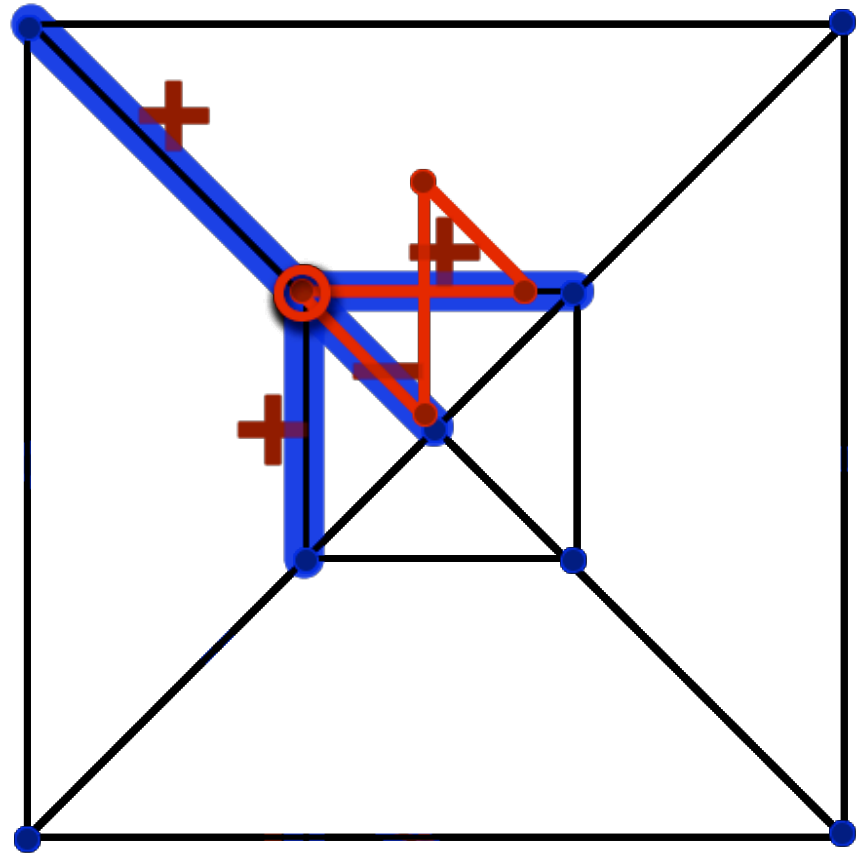




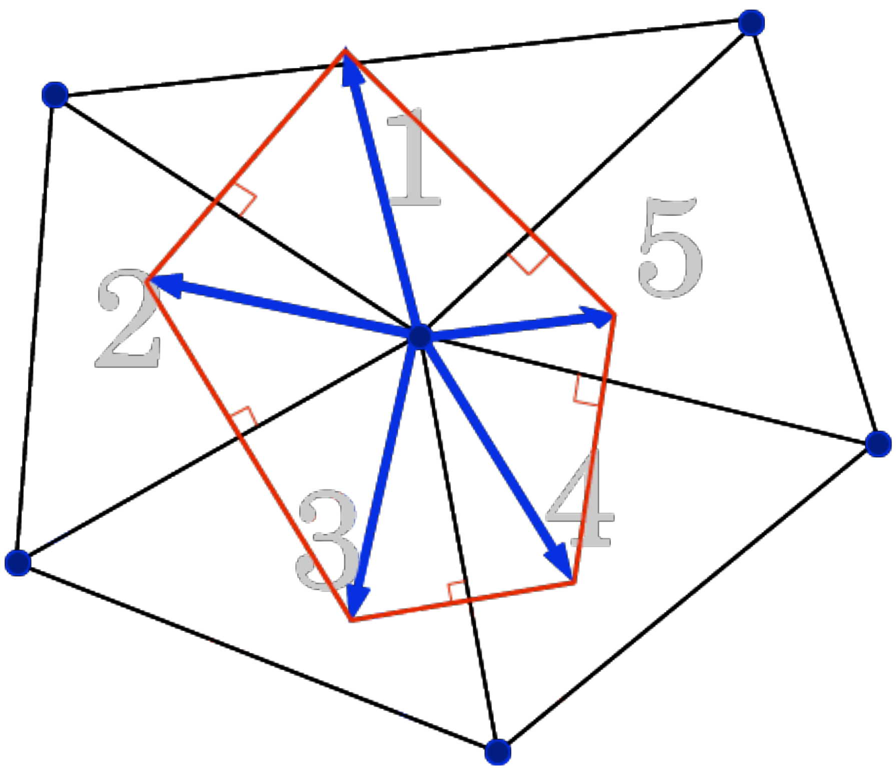
The Maxwell-Cremona Correspondence



Reciprocal Diagrams from Equilibrium Stresses



Reciprocal Diagrams from Liftings



The Maxwell-Cremona Corresondence

Equilibrium Stresses

↕

Reciprocal Diagrams

↕

Liftings

Other Famous Reciprocal Diagrams

Weighted Delaunay Triangulation

2½ dimensional polarity

Weighted Voronoi Diagram

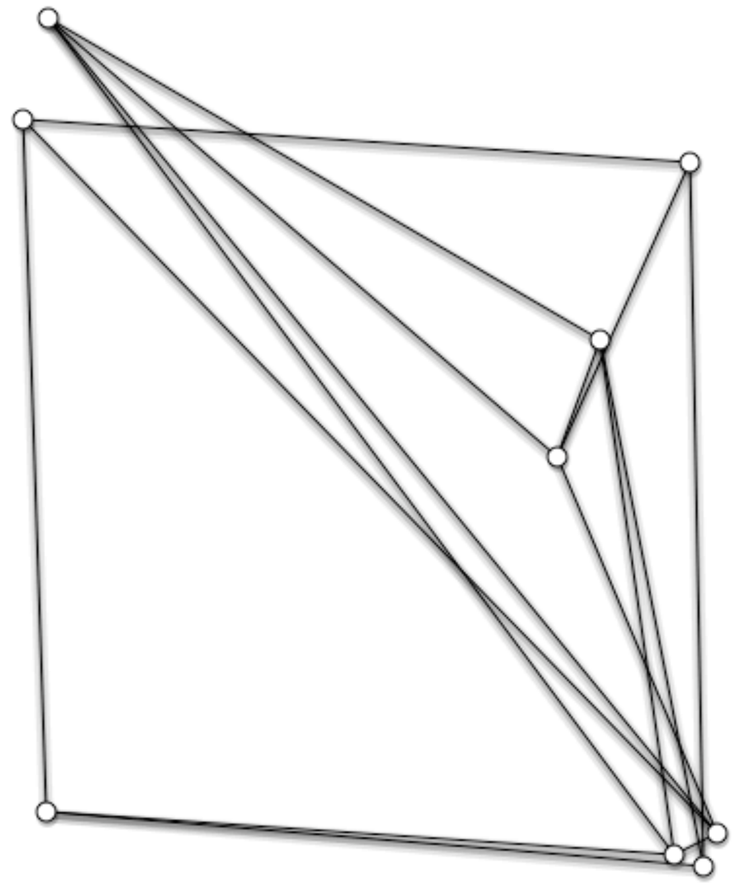
How to Draw a Graph

Tutte's Algorithm

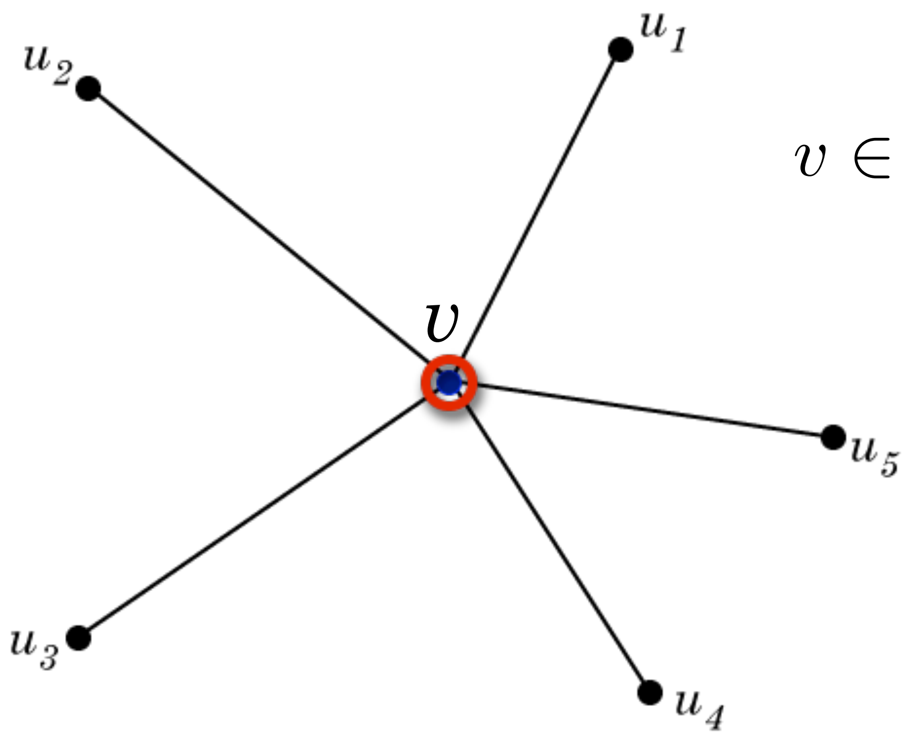
1. Fix one face of a simple, planar, 3-connected graph in convex position.
2. Place each other vertex at the barycenter (centroid) of its neighbors.

The result is a non-crossing, convex drawing.

Spring Interpretation



Computing Forces



$v \in \mathbb{R}^2$

$$F_v = \sum_{u \sim v} (v - u)$$

$$= d_v v - \sum_{u \sim v} u$$

$$F = LV$$

$$L = D - A$$

degrees *adjacency*

The Laplacian!

Computing Forces

$$LV = F = 0?$$

V_1: boundary
V_2: interior

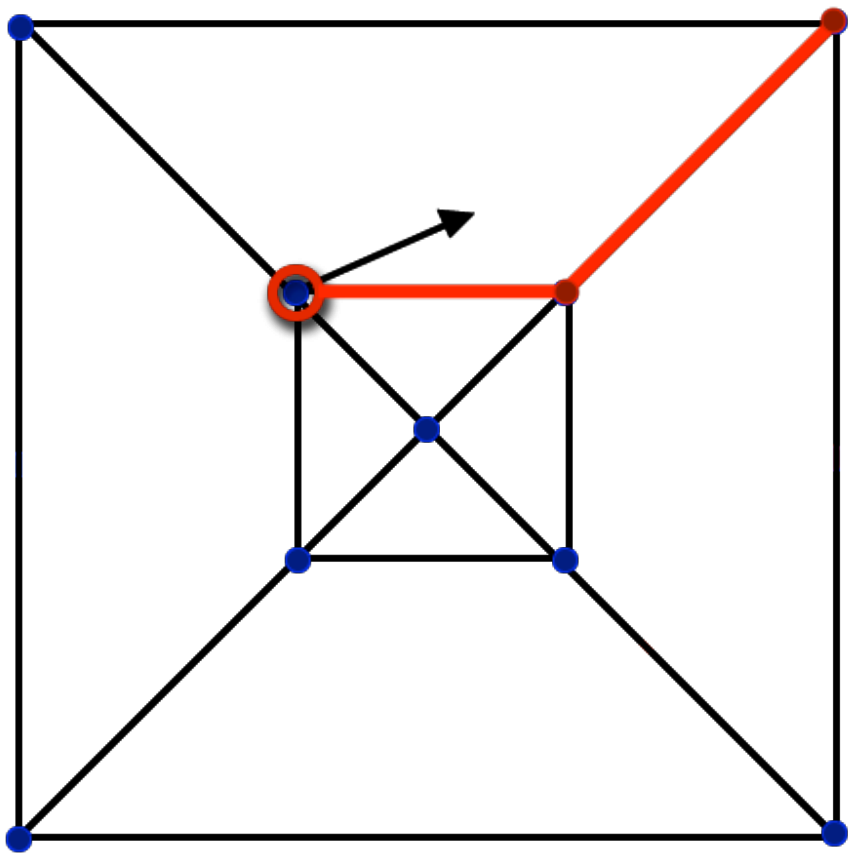
$$\begin{bmatrix} L_1 & B^T \\ B & L_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} F' \\ 0 \end{bmatrix}$$

$$BV_1 + L_2 V_2 = 0$$

$$V_2 = (-L_2^{-1} B) V_1$$

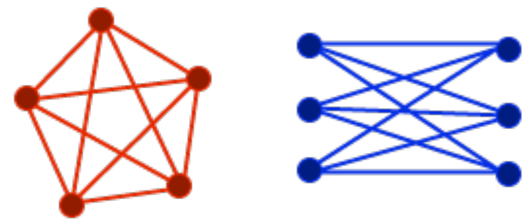
Monotone Paths

Pick a direction and a vertex. There is a monotone path in that direction from the vertex to the boundary.

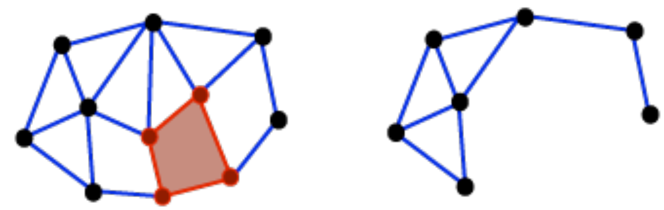


Planar, 3-Connected Graphs

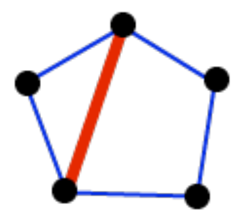
- No K_5 or $K_{3,3}$ minors



- Removing a face does not disconnect the graph.

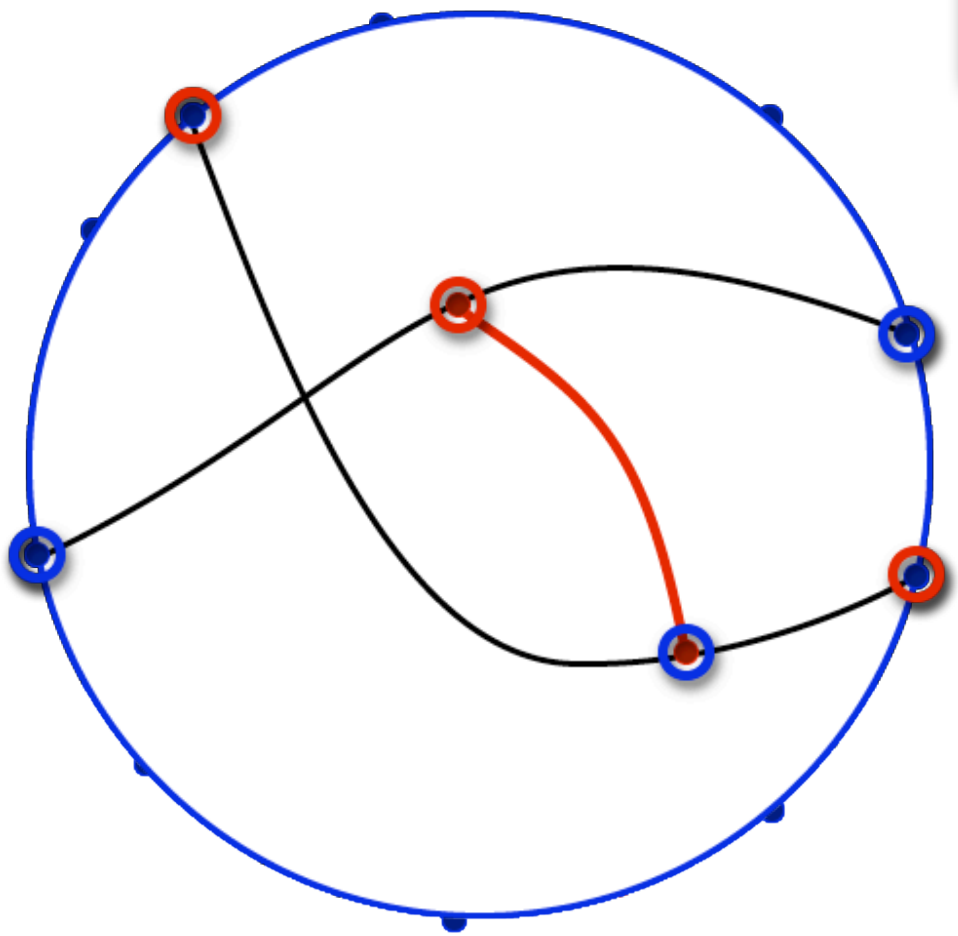


- No face has a diagonal.

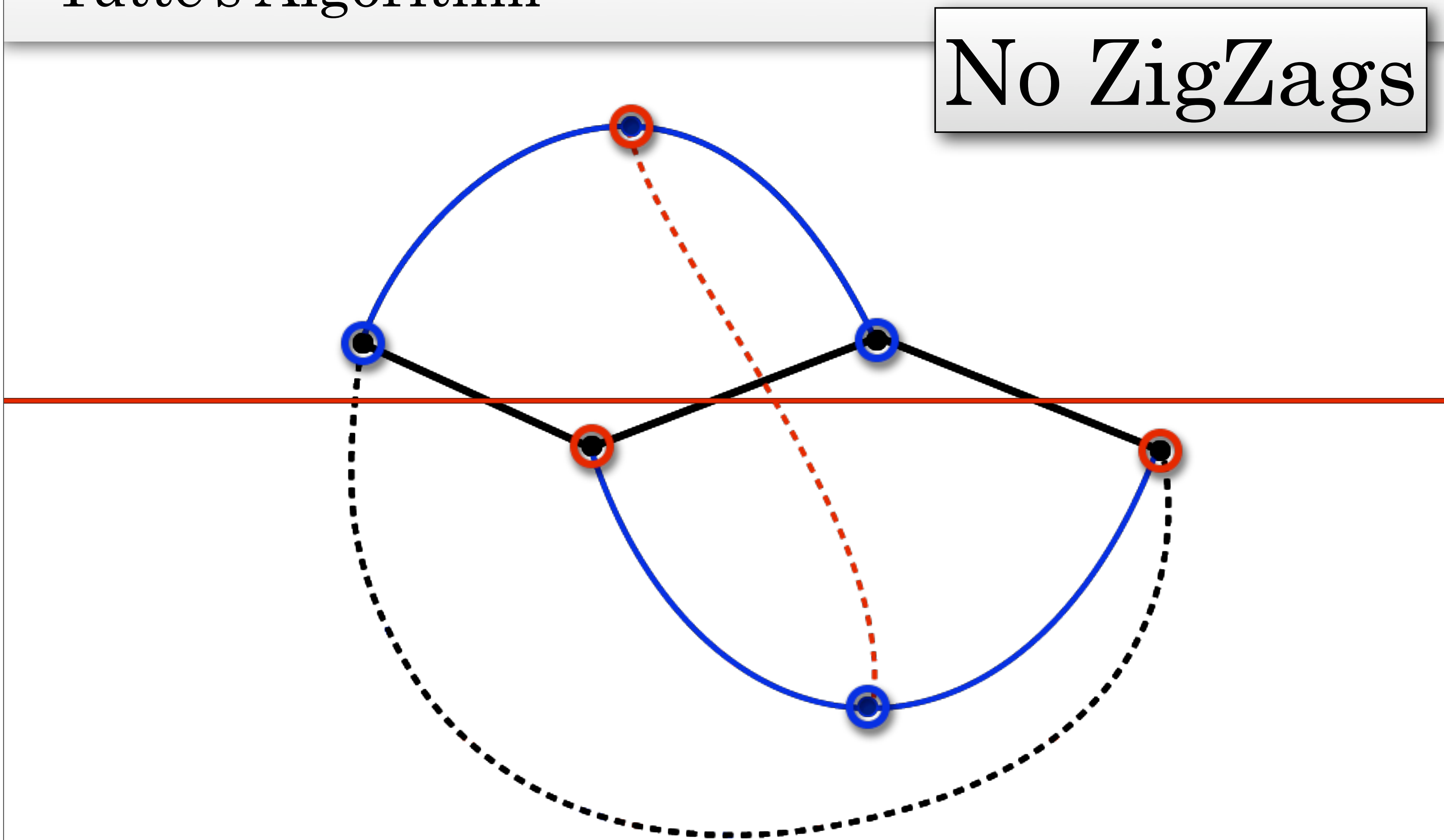


Double Crossing a Face

Lemma: No two disjoint paths have interleaved endpoints on a face.

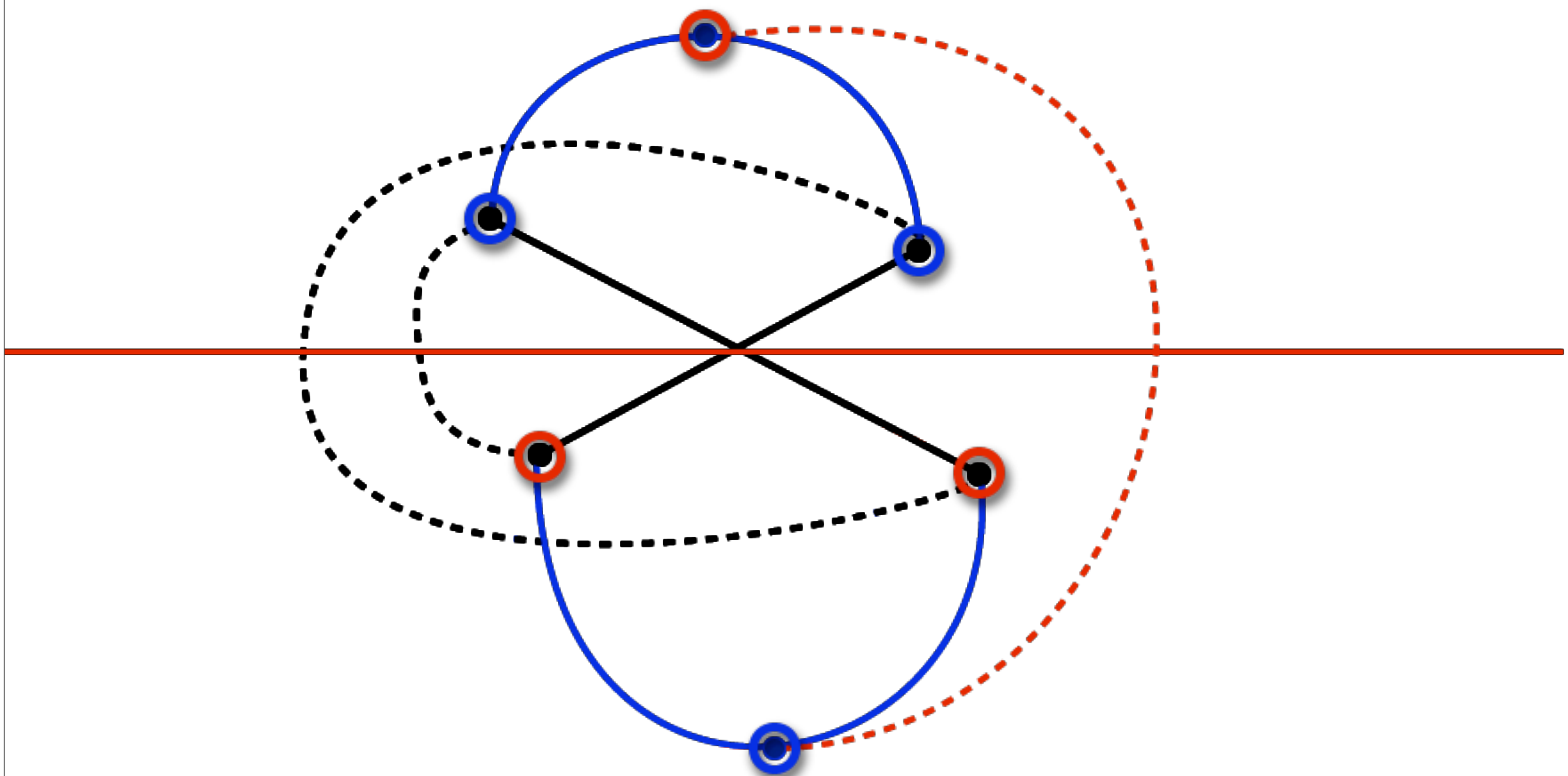


Tutte's Algorithm



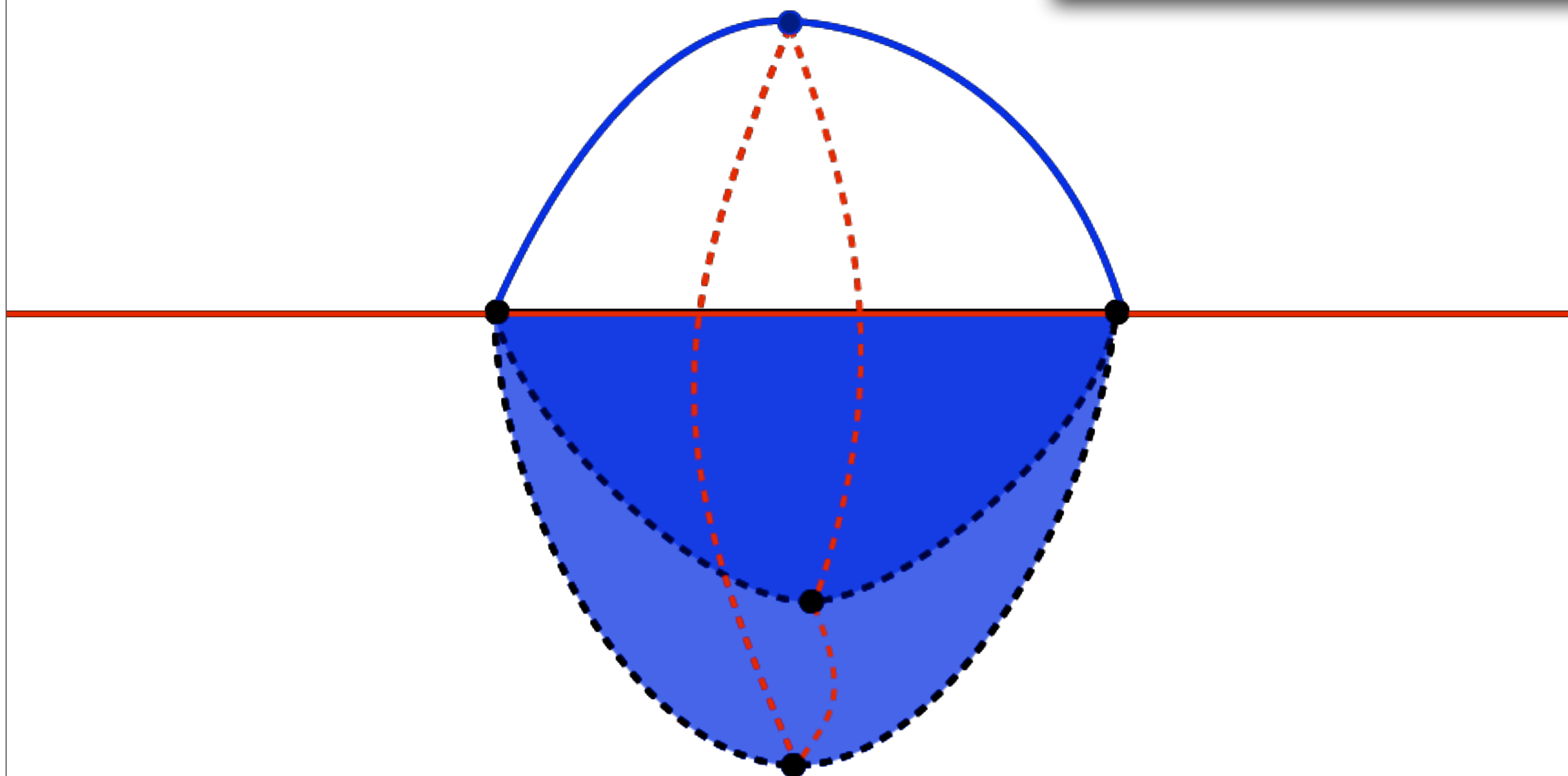
Tutte's Algorithm

No Crossings



Tutte's Algorithm

No Overlaps



Tutte and Maxwell-Cremona

- ➡ Weirdness on the outer face.
- ➡ Lifting still works, except outer face.
- ➡ Lifting is convex.

Steinitz's Theorem

Steinitz's Theorem

A graph G is the 1-skeleton of a 3-polytope if and only if it is simple, planar, and 3-connected.

Steinitz's Theorem

Claim: If the graph has a triangle, then the Tutte embedding followed by the Maxwell-Cremona lifting gives the desired polytope.

Fix the triangle as the outer face.

After the lifting, the triangle must lie on a plane.

Steinitz's Theorem

Question: What if there is no triangle?

Answer: Dualize (the dual has a triangle)

Steinitz's Theorem

Lemma: Every 3-connected, planar graph has a triangle or a vertex of degree 3.

$$|V| - |E| + |F| = 2$$

$$|E| = \frac{1}{2} \sum_{v \in V} \delta(v)$$

$$|E| = \frac{1}{2} \sum_{f \in F} |f|$$

$$\forall v \quad \delta(v) \geq 4 \Rightarrow |E| \geq 2|V|$$ (No degree 3)

$$\forall f \quad |f| \geq 4 \Rightarrow |E| \geq 2|F|$$ (No triangles)

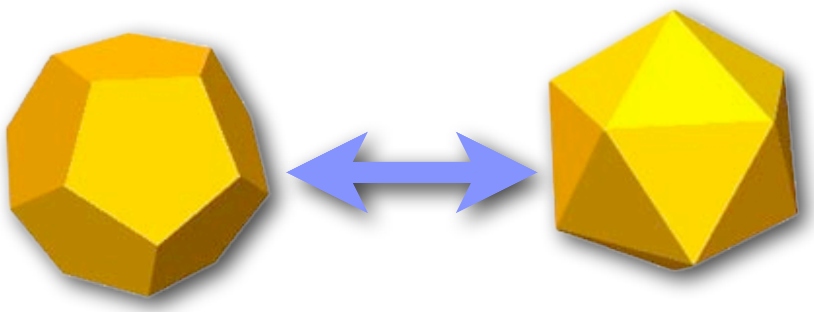
$$\frac{|E|}{2} - |E| + \frac{|E|}{2} \geq 2$$

$$0 \geq 2$$

Steinitz's Theorem

So, with the Tutte embedding and the Maxwell-Cremona Correspondence, we can construct a polytope with 1-skeleton isomorphic to *either* the graph *or* its dual.

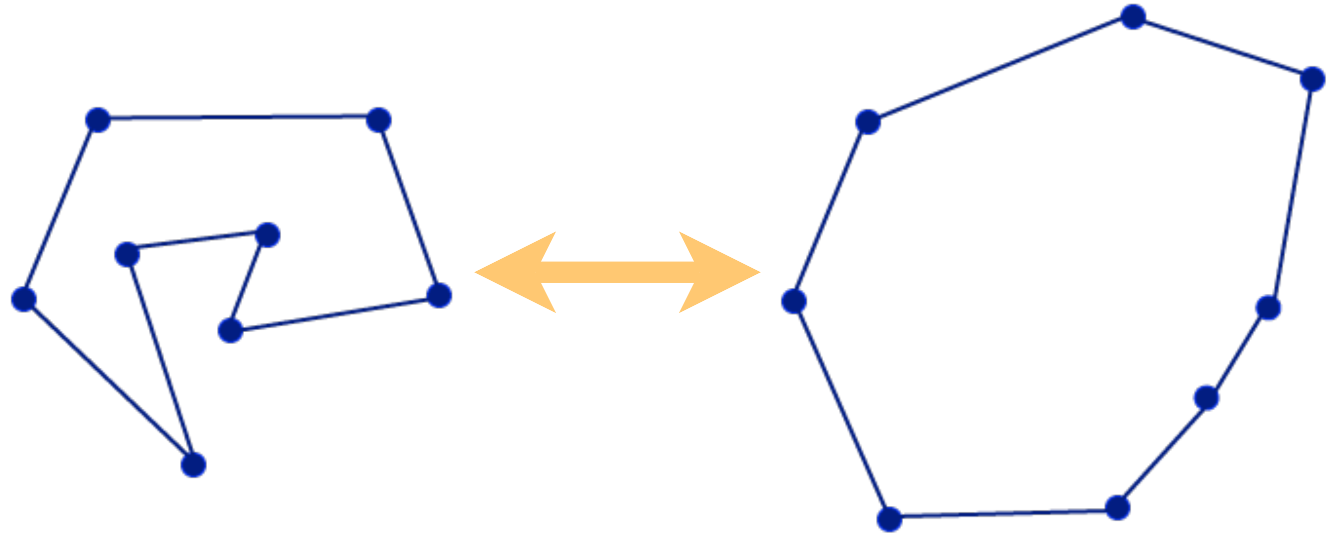
If we have the dual, *polarize*.



[Eades, Garvan 1995]

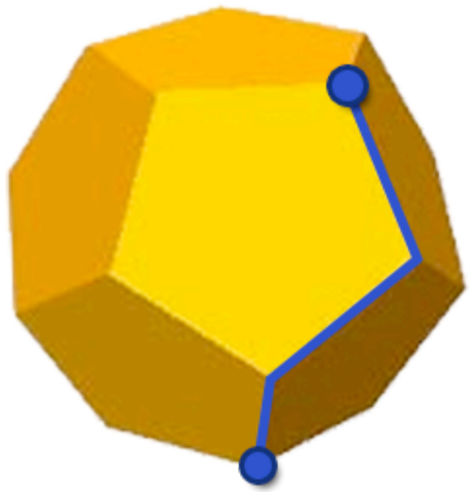
A Tour of Other Stuff

Rigidity and Unfolding



[Connelly, Demaine, Rote, 2000]

Greedy Routing

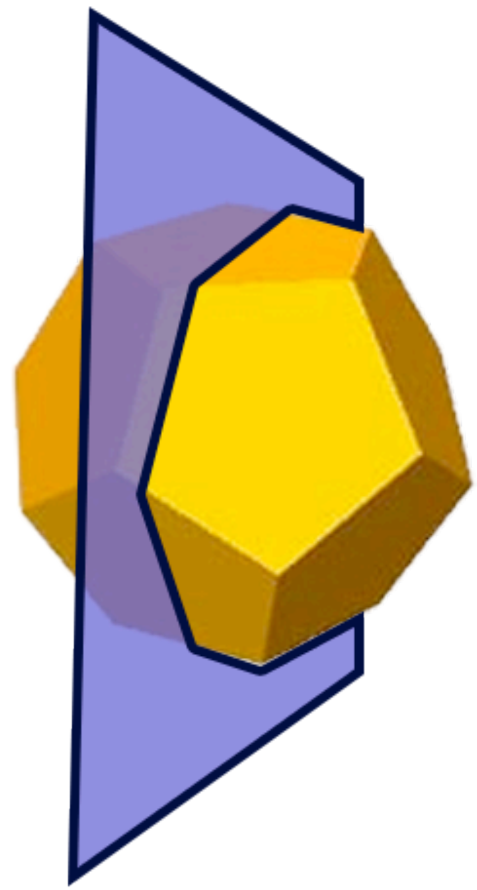


[Morin, 2001]

[Papadimitriou, Ratajczak, 2004]

Robust Geometric Computing

[Hopcroft and Kahn 1992]



Correspondence between Colin de Verdiere matrices and Steinitz representations

[Lovasz, 2000]

The construction will start with the polar polytope. Let $G^* = (V^*, E^*)$ denote the dual graph of G.

Lemma 4 *We can assign a vector w_f to each $f \in V^*$ so that whenever $ij \in E$ and fg is corresponding edge of G^*, then*

$$w_f - w_g = M_{ij}(u_i \times u_j). \tag{2}$$

Proof. Let $v_{fg} = M_{ij}(u_i \times u_j)$. It suffices to show that the vectors v_{fg} sum to 0 over the edges of any cycle in G^*. Since G^* is a planar graph, it suffices to verify this for the facets of G^*. Expressing this in terms of the edges of G, it suffices to show that

$$\sum_{j \in N(i)} M_{ij}(u_i \times u_j) = 0$$

(where, as usual, $N(i)$ denotes the set of neighbors of i). But this follows from (1) upon multiplying by u_i, taking into account that $u_i \times u_i = 0$ and $M_{ij} = 0$ for $j \notin N(i) \cup \{i\}$. □

It's Maxwell-Cremona

• • •

Thank you.

Questions?