15-456 Computational Geometry, Spring 2013

Homework 5 (60 pts) Due: 5 April 2013

Guidelines: Please justify all answers in a succienct (yet complete) manner. In particular, when presenting an algorithm the code if any should be presented at a high level. A full algorithm will contain the input, the output, and any loop invariants.

Question	Points	Score
1	25	
2	35	
Total:	60	

(25) 1. **Coloring of Knot Diagram** ($25 = 15 + 10$) A knot diagram is the projection of the knot onto the plane so that there are a finite number of crossings, each of which is transverse and no two crossings coincide; for example, see the diagrams of the trefoil knot and the figure eight knot in Figure 1. A knot diagram can be broken into *strands*, or connected components of the diagram. Each crossing in a daigram sees three strands. A *three-coloring* of a knot is a choice of white, red, or blue for each strand such that at each crossing either all three strands have the same color or all three strands have distinct colors, and at least two colors are used.

Figure 1: The knot diagram of the figure eight knot (left) and the trefoil knot (right).

(a) Show that the existence of a three-coloring is a knot invariant.

(b) Show that the trefoil knot is not equivalent to either the unknot or the figure eight knot.

(35) 2. **Discrete Fréchet Distance** ($35 = 10 + 5 + 10 + 10$) Let V be a space and let d be a metric on V. Let $P\colon [0,p] \to (V,d)$ and $Q\colon [0,q] \to (V,d)$ be polygonal curves. Let

$\text{vert}(P) = \{a_0, \ldots, a_p\}$ and let $\text{vert}(Q) = \{b_0, \ldots, b_q\}$ be the inorder traversal of the vertices of P and Q respectively. A *coupling* C is a sequence

$$(\alpha_0, \beta_0), \ldots, (\alpha_m, \beta_m)$$

with $(\alpha_i, \beta_i) \in \text{vert}(P) \times \text{vert}(Q)$, $\alpha_0 = a_0$, $\beta_0 = b_0$, $\alpha_m = a_p$, $\beta_m = b_p$, $\alpha_{i+1} \in \{a_i, 1 + a_i\}$ and $\beta_{i+1} \in \{b_i, 1 + b_i\}$. Then, we define the discrete Fréchet metric as follows:

$$d_{dF}(P, Q) = \min_C \max_{i=1,\ldots,m} d(\alpha_i, \beta_i),$$

where $\min_C$ is the minimum over all couplings between $\text{vert}(P)$ and $\text{vert}(Q)$.

(a) Show that d_{dF} is a metric.

(b) Show that $d_F \leq d_{dF}$.

(c) Let $e_{P,i}$ denote the length of the i^{th} edge in P, i.e., $e_{P,i} = d(a_{i-1}, a_i)$. Show that the following inequality holds:

$$d_{dF} \leq d_F + \max\{\max_i e_{P,i}, \max_j e_{Q,j}\}.$$

In other words, the difference between the Fréchet distance and the discrete Fréchet distnace for polygonal curves is upper bounded by the maximum edge length.

(d) Given an $O(pq)$ time algorithm to compute d_{dF}.