Computational Geometry: Homework 1 Answers

February 4, 2010

1. Let $P = \{p_1, \ldots, p_n\}$. Suppose we define

$$Q = \left\{ x \in R^d \mid x = \sum_{i=1}^{n} \alpha_i p_i, \sum_{i=1}^{n} \alpha_i = 1, \alpha_i > 0 \right\}.$$

   [Note that the last inequality is strict] Write Q in terms of $CH(P)$ and $CC(P)$.

   Answer: $Q = CC(P) \setminus CH(P)$.

   I didn't ask for a proof but if I did, something like the following would have been fine.

   First we prove that $Q \subseteq CC(P) \setminus CH(P)$. If $x \in Q$ then $x \in CC(P)$ by the definition of $CC(P)$. Suppose $x \in CH(P)$. Let S be the vertices in a facet of the convex hull containing x. If we write x as a convex combination of points in P then any points in $P \setminus S$ have coefficient 0 because they are all strictly on one side of aff(S).

   Now, we prove that converse, that $CC(P) \setminus CH(P) \subseteq Q$. For each $p_i \in P$, consider an infinite ray from $p_i \to x$. The ray intersects $CH(P)$ at some facet with vertices S. We can therefore write x as a convex combination of the points in $\{p_i\} \cup S$ as $x = \sum_{j=1}^{n} \alpha_{ij} p_j$, where $\alpha_{ij} = 0$ for any j such that $p_j \ni S \cup \{p_i\}$. Note also that $\alpha_{ii} > 0$ because $x \notin CH(P)$. Let $\alpha_i = \frac{1}{n} \sum_{j=1}^{n} \alpha_{ji}$. It is not hard to check that this is indeed an affine combination and that in fact each of the α_i's are strictly positive and therefore $x \in Q$.

2. The centroid of a point set $P \subset R^d$ is a new point in R^d whose ith coordinate is the average of the ith coordinates of the points of P. Prove that the centroid of P is contained in $CC(P)$.

   Answer: We can write the centroid as $c = \sum_{i=1}^{n} \frac{1}{n} p_i$. This combination of the p_i's is both **affine** and **non-negative** and therefore it is a convex combination.

3. Let us define a new predicate in R^2 as follows.

$$\phi(x,a,b,c) = \frac{sign \det \begin{bmatrix} x & a & b & c \\ 1 & 1 & 1 & 1 \\ |x| & |a| & |b| & |c| \end{bmatrix}}{sign \det \begin{bmatrix} a & b & c \\ 1 & 1 & 1 \end{bmatrix}}$$

What does this predicate compute about x with respect to a, b, and c? What is the significance of the denominator? What might you name this predicate? [Hint: This is just a fancy way of describing all of 8th grade math.]

Answer: The numerator lifts the points to the cone $z = +\sqrt{x^2 + y^2}$ and then performs a planeside test. The intersection of this cone and plane is a conic section focused at 0. The denominator normalizes the result so that the answer is invariant to the order of the inputs a, b, and c. A good name might be InConic.

4. Using the ccw predicate from class, write a function that tests if a point x is inside a triangle, $\triangle ABC$.

Answer: $x \in \triangle ABC \Leftrightarrow \text{CCW}(A, B, x) = \text{CCW}(B, C, x) = \text{CCW}(C, A, x)$.

5. In class, we had to delay the proof of correctness of the Graham Scan algorithm until we had seen how ccw works. Now we are ready. Prove the correctness of the Graham scan algorithm by proving the following facts.

- Prove that the output stack contains all of the vertices of the convex hull.

  Answer: Every input point gets pushed to the output so it suffices to check that no vertex of $CH(P)$ gets popped from the output. Suppose for contradiction that $v_j \in CH(P)$ and v_j was popped from the output stack. This happens only if there are points v_i, v_k such that $i < j < k$ and $\text{CCW}(v_i, v_j, v_k) = -1$, or equivalently if $\text{CCW}(v_i, v_k, v_j) = 1$. By the initial sorting, $\text{CCW}(v_0, v_i, v_j) = 1$ and $\text{CCW}(v_k, v_0, v_j) = 1$. So, using the method from the previous question, we have that $v_j \in \triangle v_0 v_i v_k$ and therefore, $x \notin CH(P)$.

- Prove that the output stack contains only the vertices of the convex hull.

  Answer: The proof is by induction. Assume that after $k-1$ points have been considered, the output stack contains $CH(v_0, \ldots, v_{k-1})$. As a base case, the hypothesis holds trivially for a single triangle. Suppose for contradiction that after $k+1$ points have been considered, the output stack contains some point not on $CH(v_0, \ldots, v_k)$. Since the output ordering is sorted, the edges cannot cross and therefore if the output points are not in convex position, there must be a sequence of three consecutive points that form a clockwise turn. By our inductive hypothesis, every three consecutive points on the output stack formed a ccw turn before adding v_k. The only "new" set of 3 consecutive points on the output stack after adding v_k is the top three points so these must form a cw turn. However, this is exactly the loop condition on the inner loop and therefore the algorithm would have popped the output before adding v_k.

- Prove that the vertices are in cyclic order.

  Answer: To be in cyclic order means that there is a point x in $CC(P)$ such that for every oriented edge $\overline{ab}$, we have $\text{CCW}(a, b,)$. It suffices to observe that $ccw(a, b, x) \geq 0$. We may set $x = v_0$ and observe that this holds for any $\overline{v_i v_j}$ with $i < j$ for the initial ordering. Because the ordering is maintained, throughout the algorithm, it holds at the end as well.

Extra Credit Recall that a regular n-gon can be constructed with a ruler and compass if and only if all of the odd prime factors of n are distinct and of the form $2^{2^k} + 1$ for any $k \in \mathbb{Z}$. Prove that this statement is still true if instead all of the odd prime factors are distinct and of the form $2^k + 1$ for any $k \in \mathbb{Z}$.

Answer: It will suffice to prove that if $2^k + 1$ is prime then k is a power of 2. Suppose for contradiction that $k = ab$ for integers a and b with b odd. We see that $2^a + 1$ divides $2^k + 1$ with a little modular arithmetic as follows.

$$2^k + 1 \equiv (2^a)^b + 1 \equiv (-1)^b + 1 \equiv -1 + 1 \equiv 0 \pmod{2^a + 1}.$$