# 15-456/852 Computational Geometry, Fall 2017

**Homework 4 (90 pts)** **Due: Dec 4 by the end of the day**

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| Question | Points | Score |
|---|---|---|
| 1 | 30 | |
| 2 | 30 | |
| 3 | 30 | |
| 4 | 30 | |
| 5 | 60 | |
| Total: | 180 | |

**Instructions:** Below please find 5 questions. If you are taking the class 15-456 you need not do question 5. If you are taking the class 15-852 then you should attempt all 5 questions.

(30) 1. **Shattering a Simplex**

1. Show that the vertices of a $d$-simplex in $\mathbb{R}^d$ can be shattered by half-spaces.
   Hint: Use the fact the vertices are affinely independent. Use linear algebra

(30) 2. **VC Dimension of Polytopes**

**Subtask 1.** Prove that the VC dimension of the set of convex polygons with at most $n$ vertices $\{CC(S) : |S| = n, S \subseteq \mathbb{R}^2\}$ is $O(n)$.

*Hint:* Think of the polygon as an intersection of halfplanes.

**Subtask 2.** Prove that the VC dimension of the set of convex 3D polytopes with at most $n$ vertices $\{CC(S) : |S| = n, S \subseteq \mathbb{R}^3\}$ is $O(n)$.

**Subtask 3.** Derive any upper bound in terms of $n$ and $d$ on the VC dimension of the set of convex $d$-dimensional polytopes with at most $n$ vertices $\{CC(S) : |S| = n, S \subseteq \mathbb{R}^d\}$.

**Open question.** Can you prove a poly$(n, d)$ bound for the previous problem?

(30) 3. **Bounding aspect ratio in a Voronoi Cell**

Suppose that $P \subset \mathbb{R}^d$ and $V$ is the Voronoi diagram of $P$, and $p \in P$. Let $V_p$ be the Voronoi cell of $p$ in $V$. For each $p \in P$ let

$$R_p = \arg\min_{r \in \mathbb{R}^+} V_p \subseteq B(p, r)$$

$$r_p = \arg\max_{r \in \mathbb{R}^+} B(p, r) \subseteq V_p$$

.

**Definition 1** *The aspect ratio of a Voronoi cell $V_p$ is $\rho_p = R_p/r_p$ and the aspect ratio of a Voronoi diagram is $\rho_P = \max_{p \in P} \rho_p$.*

Show that in $\mathbb{R}^2$, a Voronoi diagram with bounded aspect ratio implies a Delaunay triangulation with bounded aspect ratio, and vice versa.

(30) 4. **Euclidean MST**

Consider graph $G$ created by taking a 1/4-WSPD of a set of points in $\mathbb{R}^d$, and connecting the shortest edge between two sides of a well separated pair (for each well separated pair in the decomposition).

1. Show that $G$ is connected.
2. Show that $G$ contains the Euclidean minimum spanning tree on those points.
3. (Finding Euclidean MST): Suppose you have an oracle that tells you the shortest edge between two sides of a well separated pair (for any well separated pair in a well-separated pair decomposition) in $O(1)$ time. Show that you can find the Euclidean minimum spanning tree on a set of $n$ points in $\mathbb{R}^d$ in $O(n \log n)$ time, assuming constant dimension $d$. Note that this is faster than the regular MST algorithm on general graphs, which runs in $O(n \log n + E)$ time. What part of the proof breaks down if $d$ is not constant?

Afternote: Modern Euclidean MST algorithms basically follow the scheme we set up above, though they construct an approximation oracle for shortest edge rather than an exact oracle. For more information, search for Callahan and Kosaraju's paper titled *Faster algorithms for some geometric graph problems in higher dimensions.*

(60) 5. **Optimality of Ruppert's Algorithm**

For undergraduates taking the course, this question is optional. Those taking it as masters or graduate students should try out this problem.

In class we showed that Ruppert's Delaunay refinement algorithm terminated with all angle at least $20^o$. The goal of this exercise is to show that up to constants it will generate an optimal size such mesh. We will do this by showing that Ruppert's Algorithm and any other optimal mesh has a nice characterization.

For simplicity, in this problem we will only be concerned with meshing sets of points, without line segments. However, a similar result also holds for planar straight line graphs. First, let us state the problem formally.

**Input.** A set of points $S \subseteq \mathcal{B}$, all of which are contained in some square bounding box $\mathcal{B}$. We assume the corners of $\mathcal{B}$ belong to $S$.

**Output.** A set $T \supseteq S$ such that the Delaunay triangulation of $T$ has no angles less than $15°$.

We want to show that the output of Ruppert's algorithm is a set of size at most a constant factor greater than optimal possible.

Let us recall the core tool we use to analyze meshings. The *local feature size*, $lfs_A(x)$ for a point $x \in \mathcal{B}$ with respect to a finite set $A \subseteq \mathcal{B}$ is simply the distance to the second nearest neighbor in $A$ to $x$.

We have seen in class that the output $R$ of Ruppert's algorithm guarantees

$$lfs_R(x) \geq c \cdot lfs_S(x) \text{ for all } x \in \mathcal{B} \tag{1}$$

for some constant $c_0$.

First, we will want to show that

$$|R| \leq c_1 \int_{x \in \mathcal{B}} \frac{1}{lfs_S(x)^2} dx \tag{2}$$

for some constant $c_1$.

Recall that problem 3, we showed that the the cells of the Voronoi diagram of any correct output $T$ to our problem have constant aspect ratio (that is, the distance from the center is within a constant factor for all the points on the boundary). It is equivalent to showing that the ratio of out-radius to in-radius is bounded.

**Subtask 1.** Conclude that the Voronoi cell around each $p \in R$ contains a ball of radius $\Omega(lfs_R(p))$. Show that the integral of $lfs_R(x)^{-2}$ over this ball is constant. Use this and Inequality 1 to prove Inequality 2.

*Hint*: Use that the *lfs* function is 1-Lipschitz.

Now, let $T$ be any superset of $S$ whose Delaunay triangulation has no angles less than $15°$. We want to show that

$$|T| \geq c_2 \int_{x \in \mathcal{B}} \frac{1}{lfs_S(x)^2} dx \tag{3}$$

for some constant $c_2$.

**Subtask 2.** Show that if $x$ belongs to the Voronoi cell of some $p \in T$, then

$$lfs_T(x) = \Omega(lfs_T(p)).$$

**Subtask 3.** Use this to derive Inequality 3.

*Hint 1*: Use the result of Subtask 1.

*Hint 2*: What is the relation between $lfs_S(x)$ and $lfs_T(x)$?

**Subtask 4.** Conclude that $|T| = \Omega(|R|)$.