Lecture 26: The Fast Fourier Transform

David Woodruff

Thanks to Ryan O' Donnell for many slides

Polynomial multiplication

Let $P(x)$ and $Q(x)$ be polynomials of degree $< N$.

Assumed in "Coefficients Representation",

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{N-1} x^{N-1}$$
$$Q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1} x^{N-1}$$

Let $R(x) = P(x) \cdot Q(x)$, of degree $< 2N$.

Task is to get $R(x)$ in Coefficients Representation.

Naively: takes $O(N^2)$ time to compute $R(x)$

Polynomial multiplication

Let $P(x)$ and $Q(x)$ be polynomials of degree $< N$.

Assumed in "Coefficients Representation",

Let $R(x) = P(x) \cdot Q(x)$, of degree $< 2N$.

Task is to get $R(x)$ in Coefficients Representation.

If only everything were in "Values Representation" instead…

Polynomial multiplication

Let $P(x)$ and $Q(x)$ be polynomials of degree $< N$.

Assumed in “Coefficients Representation”,

Let $R(x) = P(x) \cdot Q(x)$, of degree $< 2N$.

Task is to get $R(x)$ in Coefficients Representation.

If only we knew

$P(1), P(2), \ldots, P(2N)$,

$Q(1), Q(2), \ldots, Q(2N)$,

$R(1), R(2), \ldots, R(2N)$

multiply in O(N) time

uniquely determines $R(x)$ by interpolation

A Divide and Conquer Approach

Want to evaluate P(x) at $x_1, x_2, \ldots, x_{2N}$ in O(N log N) time

Write P(x) = $P^0(x^2) + x\,P^1(x^2)$, where

$P^0(x)$ contains the even terms and $x \cdot P^1(x)$ contains the odd terms

Example: $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$

$P^0(x) = a_0 + a_2x + a_4x^2$

$P^1(x) = a_1 + a_3x + a_5x^2$

Why is this useful?

A Divide and Conquer Approach

Want to evaluate P(x) at $x_1, x_2, \ldots, x_{2N}$ in O(N log N) time

Write P(x) = $P^0(x^2) + x\,P^1(x^2)$, where

$P^0(x)$ contains the even terms and $x \cdot P^1(x)$ contains the odd terms.

If my points are $x_1, x_2, \ldots, x_N, -x_1, -x_2, \ldots, -x_N$,

I just need the evaluations of $P^0(x)$ and $P^1(x)$ at $x_1^2, \ldots, x_N^2$

$T(2N) = 2T(N) + O(N)$ with solution T(2N) = O(N log N), are we done?

We need points that can be recursively partitioned into +/-

Use the Complex Roots of Unity

Write $P(x) = P^0(x^2) + x\,P^1(x^2)$, where

$P^0(x)$ contains the even terms and $x \cdot P^1(x)$ contains the odd terms.

Choose 2N points to be the complex 2N-th roots of unity

Key fact: the 2N squares of the 2N-th roots of unity are:

first the N N-th roots of unity, then again the N N-th roots of unity

$T(2N) = 2T(N) + O(N)$ with solution $T(2N) = O(N \log N)$!!

What are the complex N-th roots of unity?

Discrete Fourier Transform (& Inverse)

Let N be a power of 2.

$S_N = \{1, \omega_N^1, \omega_N^2, \omega_N^3, \ldots, \omega_N^{N-1}\}$ is the set of N "complex roots of unity" that I'll describe shortly.

Let P(x) be a polynomial of degree N−1.

P's coefficients $\xrightarrow[\text{evaluation}]{\mathbf{DFT_N}}$ P's values on S_N

P's values on S_N $\xrightarrow[\text{interpolation}]{\mathbf{IDFT_N}}$ P's coefficients

Fast Fourier Transform

A recursive algorithm for DFT_N and $IDFT_N$ that uses only $O(N \log N)$ arithmetic operations.

P's coefficients $\xrightarrow[\text{evaluation}]{\mathbf{DFT_N}}$ P's values on S_N

P's values on S_N $\xrightarrow[\text{interpolation}]{\mathbf{IDFT_N}}$ P's coefficients

Multiplying polynomials with the FFT

Let P(x), Q(x) be polynomials of degree < N.

Want R(x) = P(x)·Q(x), which has degree < 2N.

1. Use DFT_{2N} to get P(w), Q(w) for all $w \in S_{2N}$
2. Multiply pairs, getting R(w) for all $w \in S_{2N}$
3. Use $IDFT_{2N}$ to get R's coefficients

P's coefficients —**DFT_N** (evaluation)→ P's values on S_N

P's values on S_N —**$IDFT_N$** (interpolation)→ P's coefficients

Multiplying polynomials with the FFT

Let P(x), Q(x) be polynomials of degree < N.

Want R(x) = P(x)·Q(x), which has degree < 2N.

1. Use DFT_{2N} to get P(w), Q(w) for all $w \in S_{2N}$
2. Multiply pairs, getting R(w) for all $w \in S_{2N}$
3. Use $IDFT_{2N}$ to get R's coefficients

Time:

1. O(N log N) arithmetic ops
2. O(N) arithmetic ops
3. O(N log N) arithmetic ops

O(N log N) arithmetic ops

Multiplying polynomials with the FFT

Can multiply two degree-N polynomials using O(N log N) arithmetic operations.

If each coefficient is a word of O(log N) bits, can multiply the polynomials in O(N log N) time.

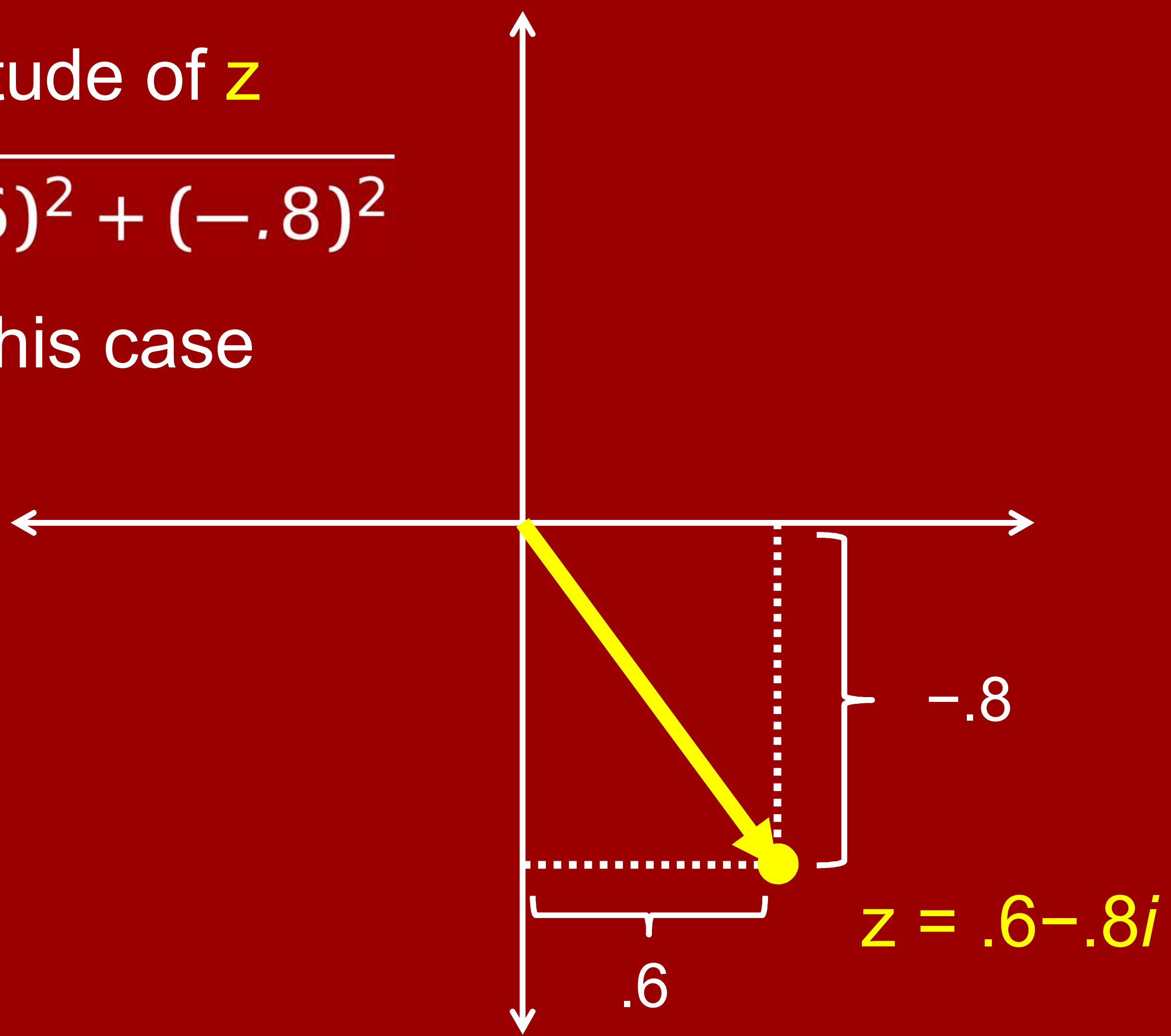
* Requires proving that you can compute the N^{th} roots of unity to O(log N) bits of precision in O(N log N) time, and that this precision is sufficient. This is fairly easy to prove, but also boring to prove.

The complex numbers $\mathbb{C}$

$|z|$ = magnitude of z

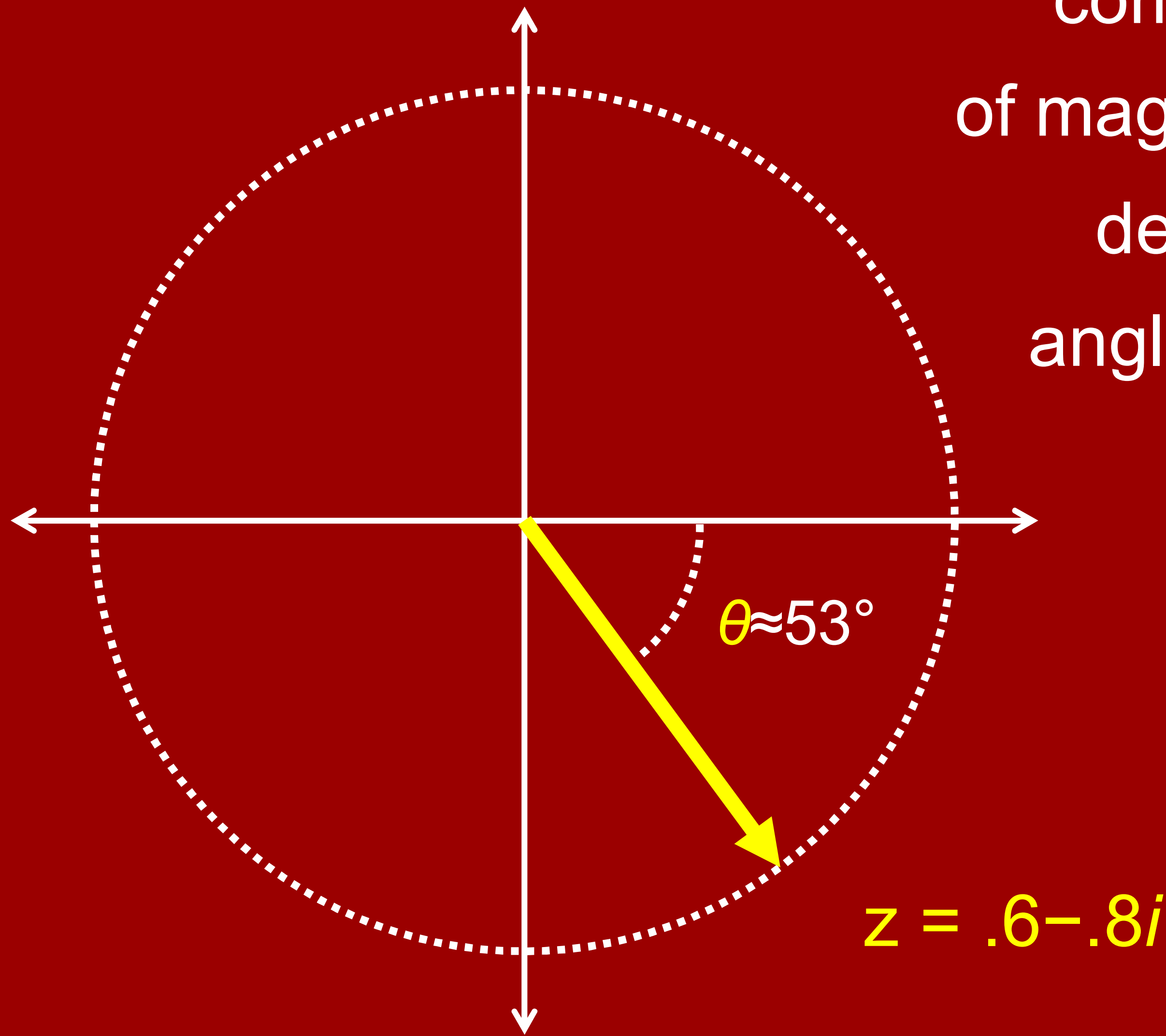
$$= \sqrt{(.6)^2 + (-.8)^2}$$

= 1, in this case



The complex numbers ℂ

complex #'s of magnitude 1 defined by angle θ from x-axis

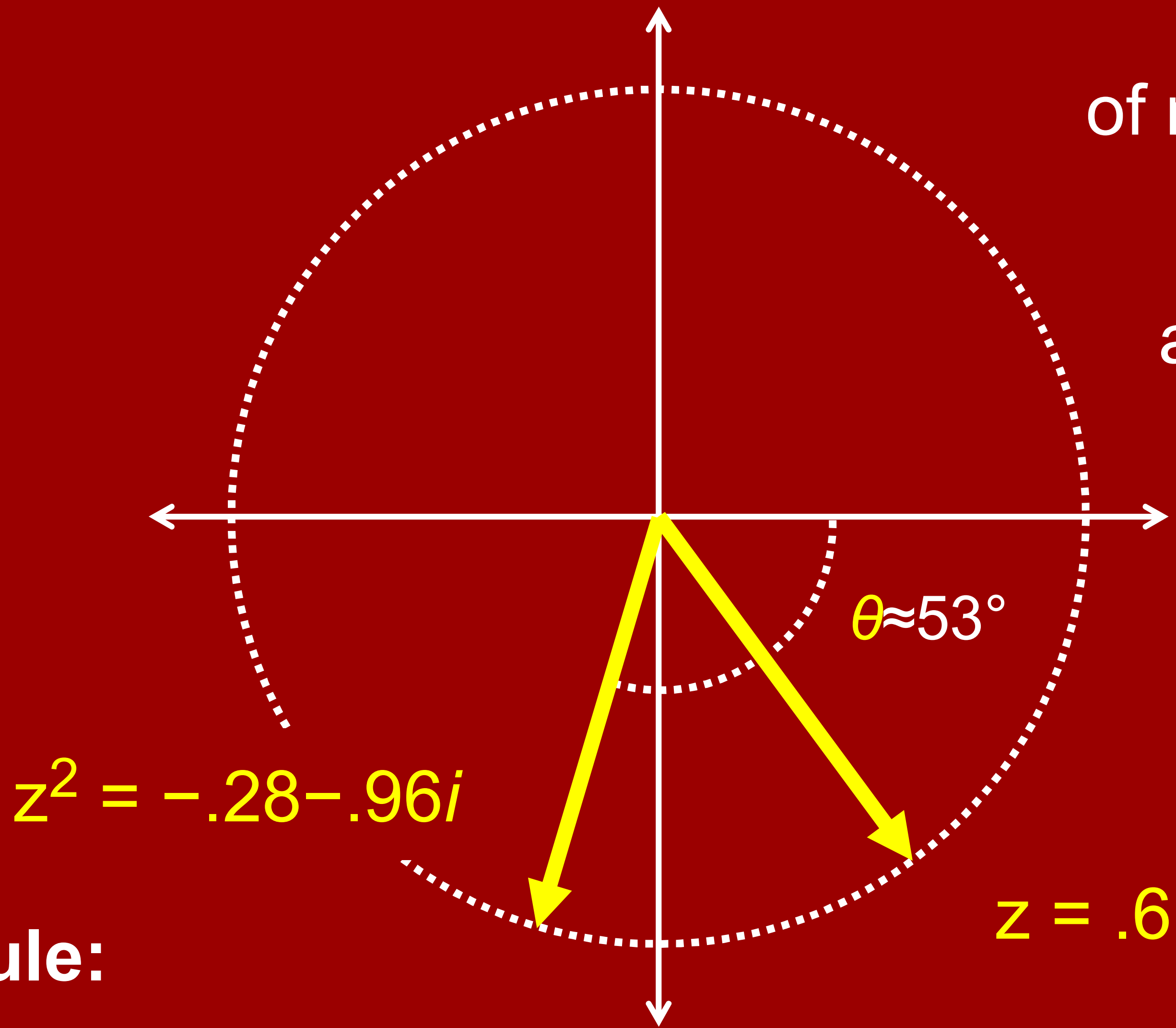


Key Rule:

Multiplication by z = rotation by θ.

The complex numbers ℂ

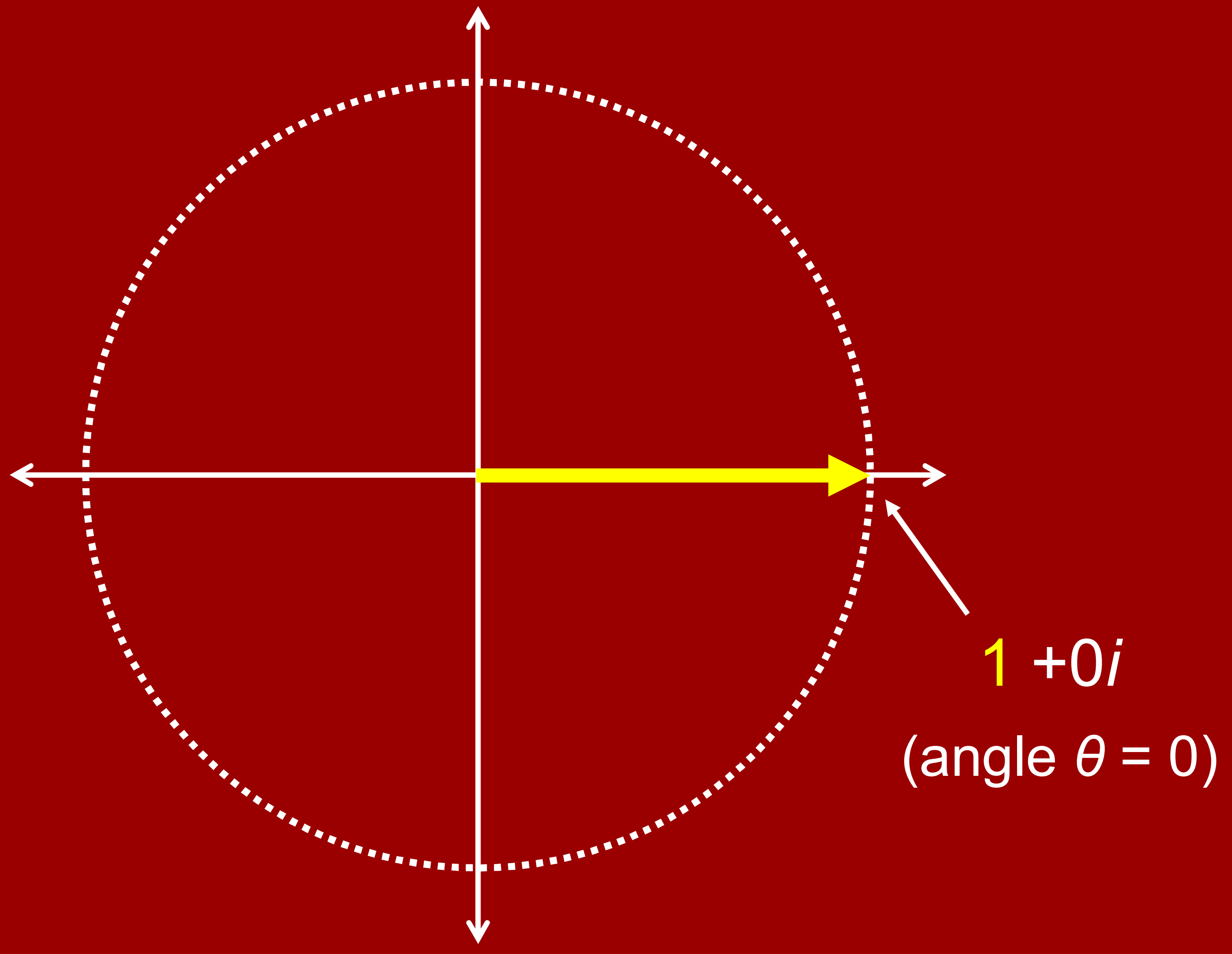
complex #'s of magnitude 1 defined by angle θ from x-axis



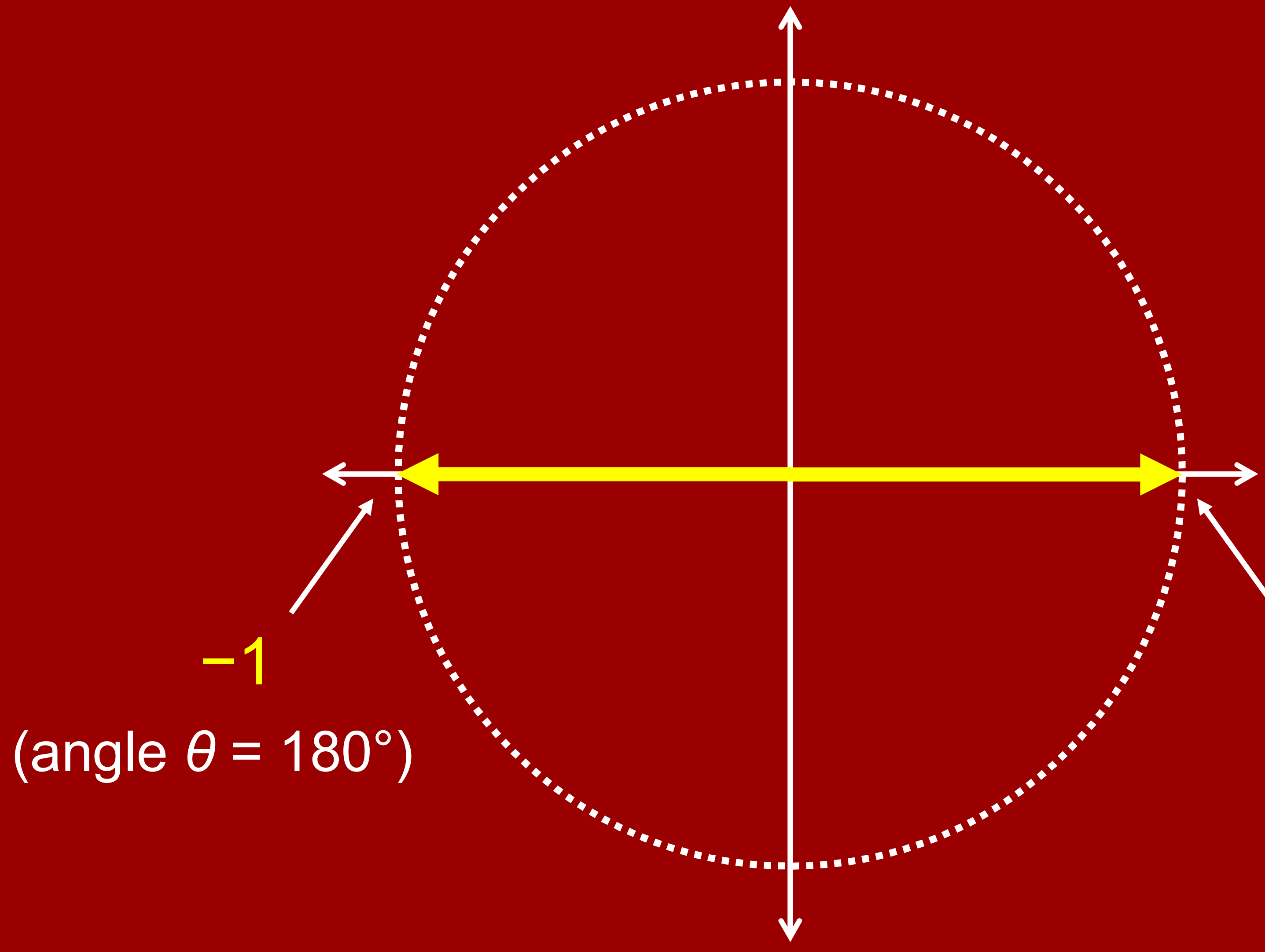
Key Rule:

Multiplication by z = rotation by θ.

Unity

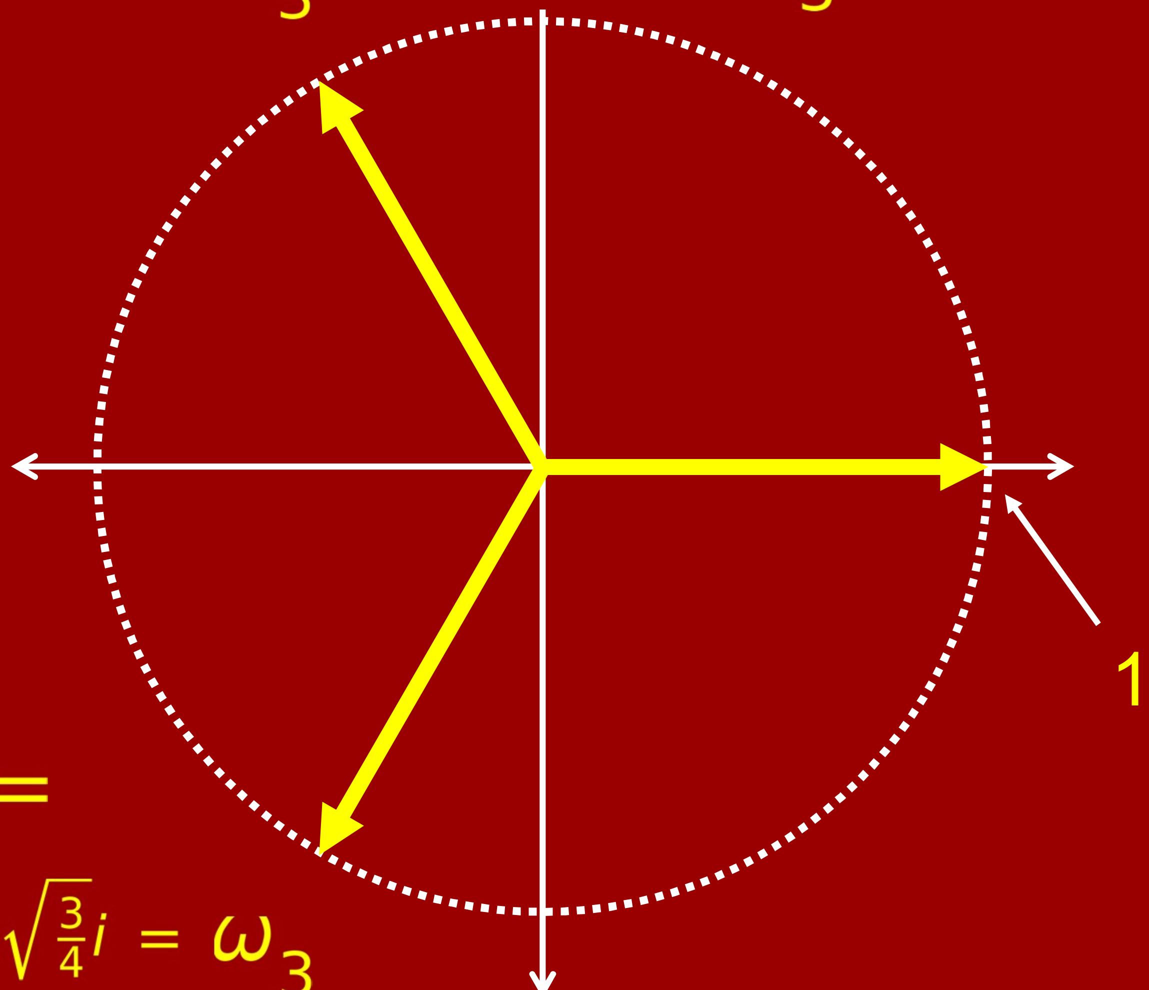


Square Roots of Unity



Cube Roots of Unity

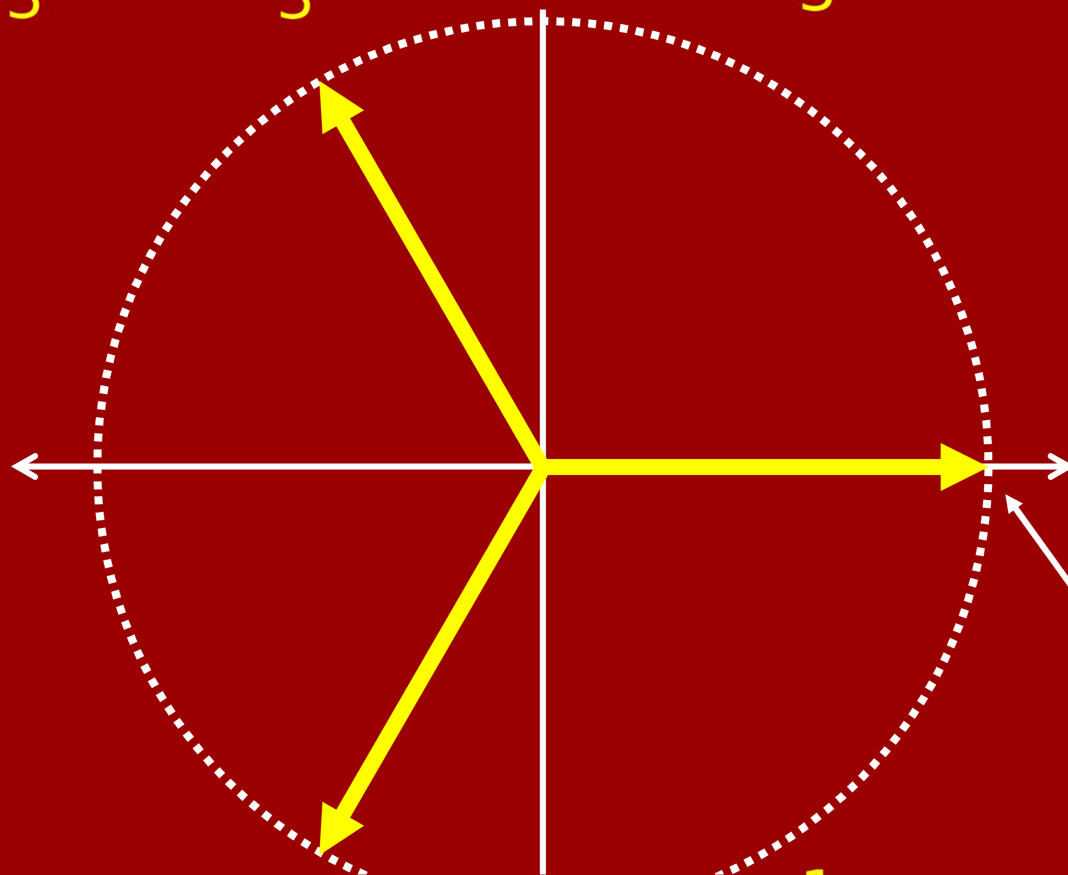
ω_3^2 = rotation by $\frac{2}{3}$ of a circle



$$e^{\frac{-2\pi i}{3}} = -\sqrt{\frac{1}{4}} - \sqrt{\frac{3}{4}}i = \omega_3$$

Cube Roots of Unity

$\omega_3^{-1} = \omega_3^2$ = rotation by $\frac{2}{3}$ of a circle

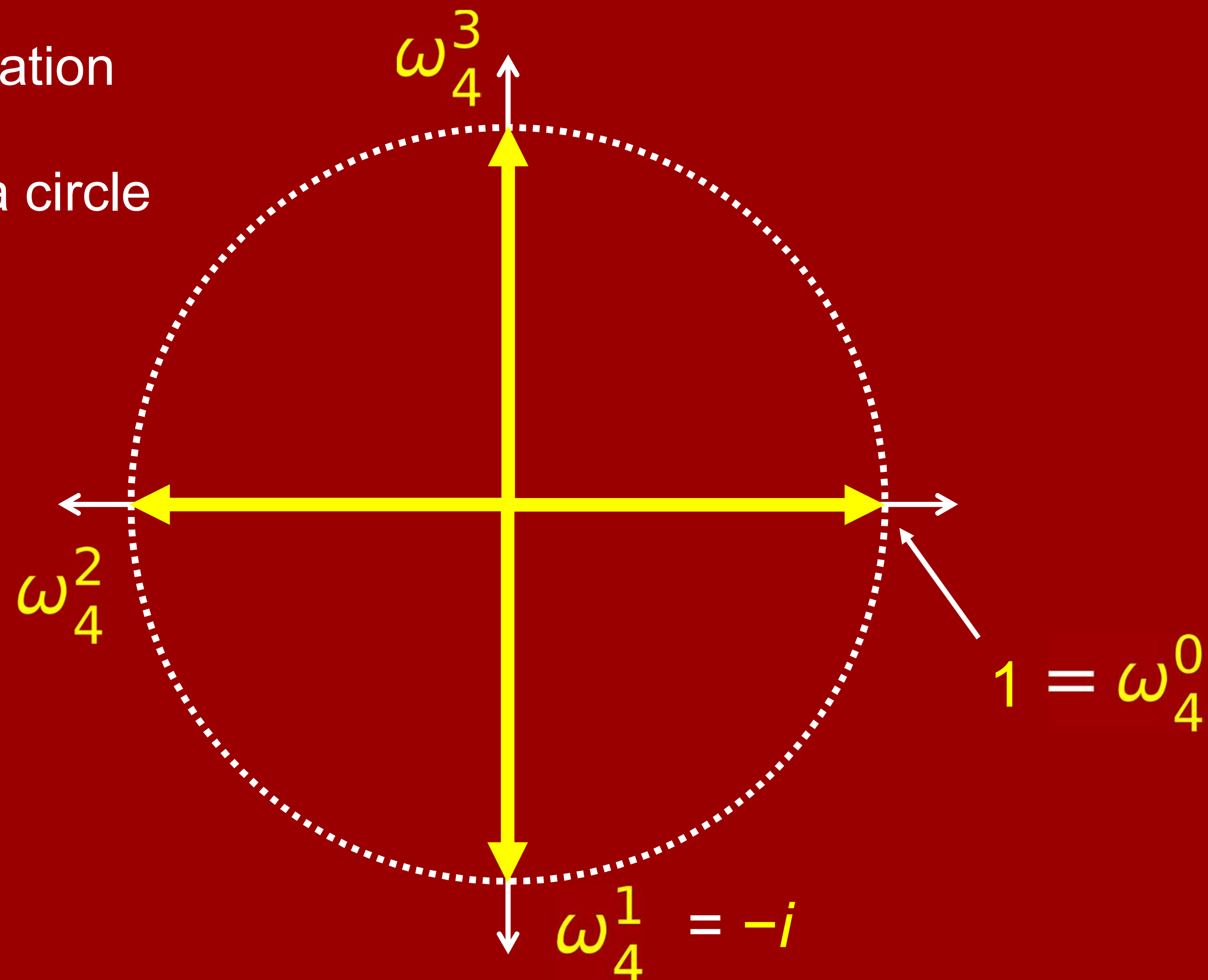


$1 = \omega_3^3 = \omega_3^0$

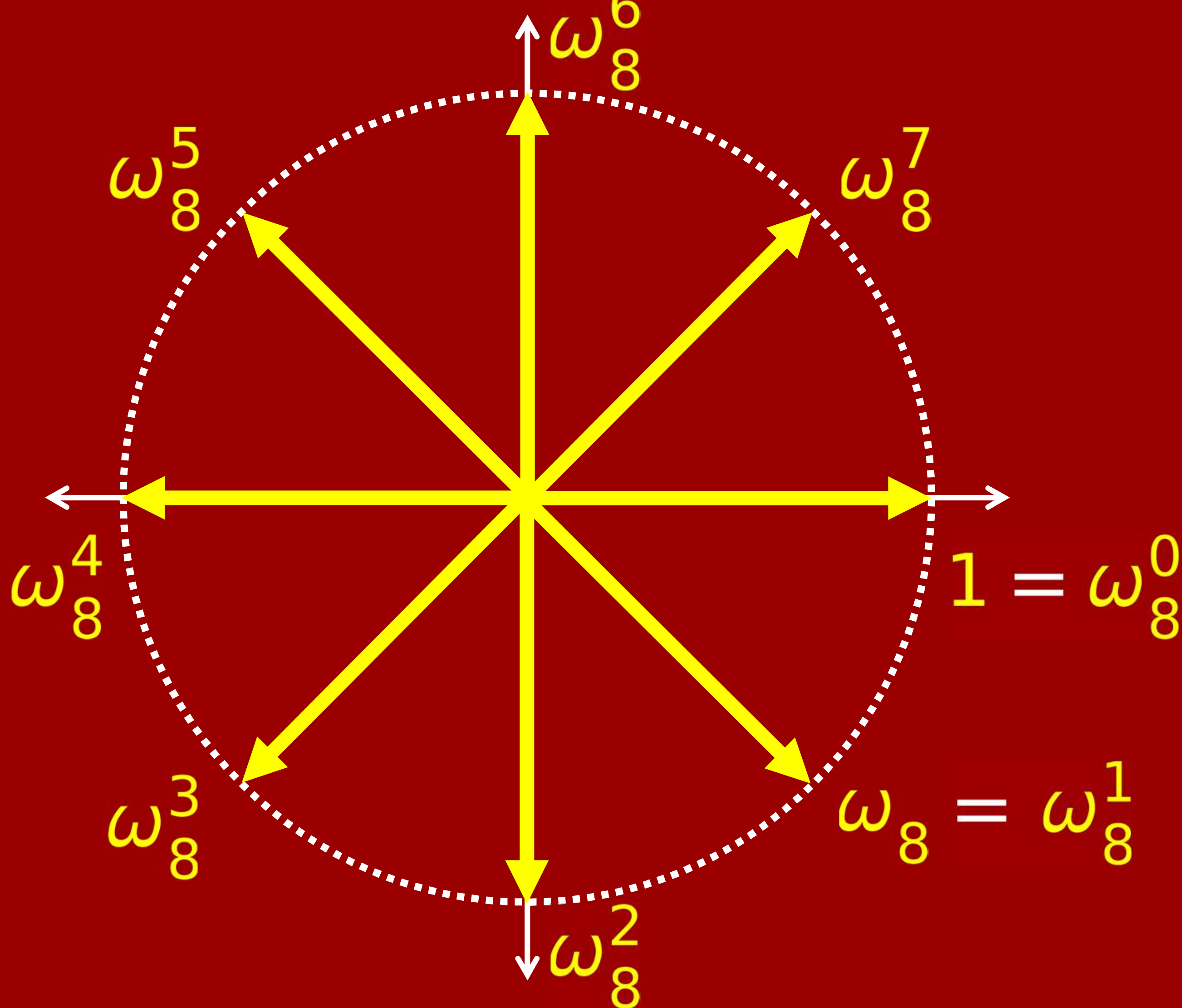
ω_3^1 = rotation by $\frac{1}{3}$ of a circle

4^{th} Roots of Unity

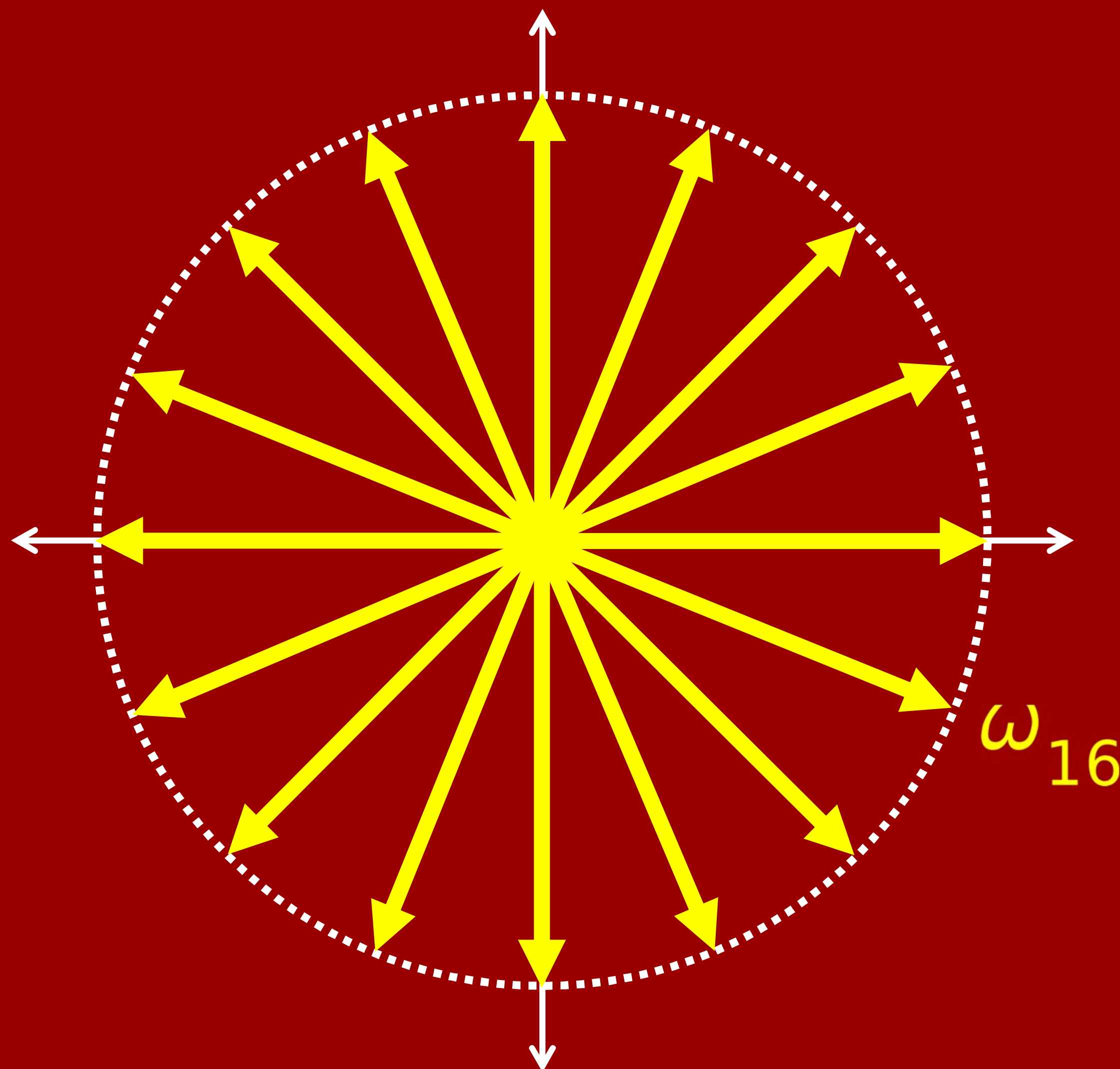
ω_4^j = rotation by $\frac{j}{4}$ of a circle



8^{th} Roots of Unity



16^{th} Roots of Unity



Discrete Fourier Transform (& Inverse)

Let N be a power of 2.

Let $S_N = \{1, \omega_N^1, \omega_N^2, \omega_N^3, \ldots, \omega_N^{N-1}\}$

Let P(x) be a polynomial of degree N−1.

P's coefficients $\xrightarrow[\text{evaluation}]{\mathbf{DFT_N}}$ P's values on S_N

P's values on S_N $\xrightarrow[\text{interpolation}]{\mathbf{IDFT_N}}$ P's coefficients

Discrete Fourier Transform (& Inverse)

Let N be 8, and let $\omega = \omega_8$

Let $S_8 = \{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7\}$

Let P(x) be a polynomial of degree 7.

P's coefficients $\xrightarrow[\text{evaluation}]{\mathbf{DFT_8}}$ P's values on S_8

P's values on S_8 $\xrightarrow[\text{interpolation}]{\mathbf{IDFT_8}}$ P's coefficients

Evaluation at $\{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7\}$

Say $P(x) = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ 1 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

Since $\omega^8 = 1$, we can reduce all exponents mod 8.

Evaluation at $\{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7\}$

Say $P(x) = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

DFT_8

$DFT_8[j,k] = \omega^{jk \bmod 8}$ $(0 \le j, k \le 7)$

$\ldots{}^4, \omega^5, \omega^6, \omega^7\}$

$\ldots x^6 + a_7 x^7$

Multiplication modulo 8 table

•	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

$DFT_8[j,k] = \omega^{jk \bmod 8}$ $(0 \le j, k \le 7)$

Evaluation at $\{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7\}$

Say $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

$DFT_8[j,k] = \omega^{jk \bmod 8}$ $(0 \le j, k \le 7)$

Evaluation at $\{1, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7\}$

Say $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$

$$DFT_8 \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

Interpolation?

Say $P(x) = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7$.

Given $P(1)$, $P(\omega)$, …, $P(\omega^7)$, how to get a_0, a_1, …, a_7?

$$\mathrm{DFT}_8 \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

Interpolation?

Say $P(x) = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7$.

Given $P(1)$, $P(\omega)$, …, $P(\omega^7)$, how to get a_0, a_1, …, a_7?

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \mathrm{DFT}_8^{-1} \cdot \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

also known as

IDFT_8

P's coefficients $\xrightarrow[\text{evaluation}]{\mathbf{DFT_N}}$ P's values on S_N

P's values on S_N $\xrightarrow[\text{interpolation}]{\mathbf{IDFT_N}}$ P's coefficients

also known as $IDFT_8$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = DFT_8^{-1} \cdot \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

Interpolation?

Say $P(x) = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7$.

Given $P(1), P(\omega), \ldots, P(\omega^7)$, how to get $a_0, a_1, \ldots, a_7$?

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = DFT_8^{-1} \cdot \begin{bmatrix} P(1) \\ P(\omega) \\ P(\omega^2) \\ P(\omega^3) \\ P(\omega^4) \\ P(\omega^5) \\ P(\omega^6) \\ P(\omega^7) \end{bmatrix}$$

also known as $IDFT_8$

DFT versus IDFT

Question:

We know what matrix DFT_8 is.

What is its inverse matrix, $IDFT_8$?

Answer:

It's extremely similar to DFT_8.

IDFT_8 $\qquad\qquad$ DFT_8

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

$$\text{DFT}_N[j,k] = \omega^{jk \bmod N}$$

$(0 \leq j, k < N, \ \omega = \omega_N$ is N^{th} root of unity$)$

$$\text{IDFT}_N[j,k] = \frac{1}{N}\omega^{-jk \bmod N}$$

$$\text{IDFT}_8 \qquad\qquad \text{DFT}_8$$

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

Proof illustration.

We'll show the product =

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{IDFT_8}{\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix}} \underset{DFT_8}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{IDFT}_8 = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix}$$

$$\text{DFT}_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{IDFT}_8 \qquad\qquad \mathrm{DFT}_8$$

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{IDFT}_8 \quad\quad\quad\quad \text{DFT}_8$$

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

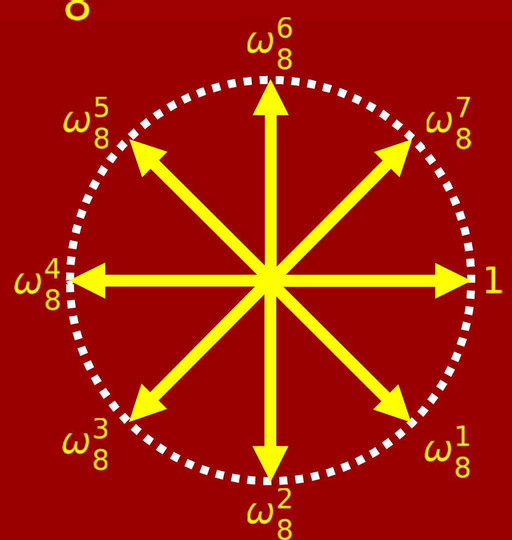
IDFT_8 DFT_8

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1+\omega^1+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6+\omega^7}{8}$$

average is 0

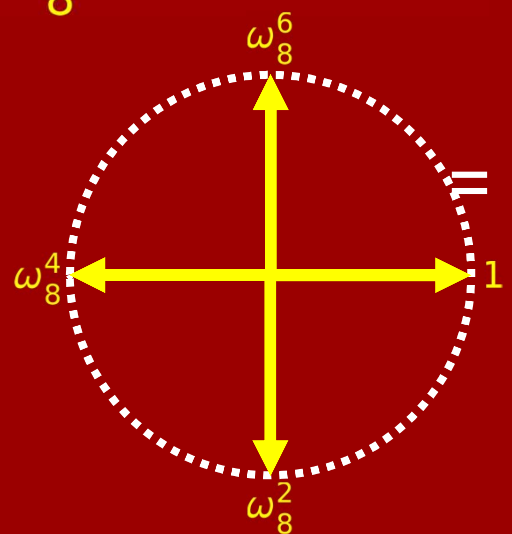


$$
\mathrm{IDFT}_8 \qquad\qquad\qquad\qquad \mathrm{DFT}_8
$$

$$
\frac{1}{8}\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\
1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\
1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\
1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\
1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\
1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\
1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\
1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\
1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\
1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\
1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\
1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\
1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega
\end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\frac{1 + \omega^2 + \omega^4 + \omega^6 + 1 + \omega^2 + \omega^4 + \omega^6}{8}
$$

average is 0

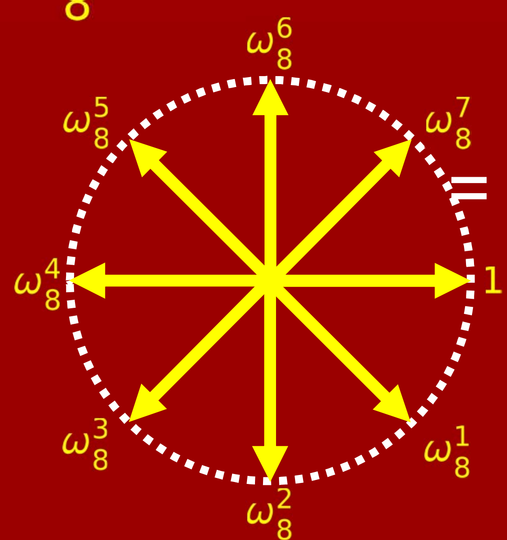


$$\text{IDFT}_8 \qquad\qquad\qquad\qquad \text{DFT}_8$$

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1 + \omega^3 + \omega^6 + \omega^1 + \omega^4 + \omega^7 + \omega^2 + \omega^5}{8}$$

average is 0



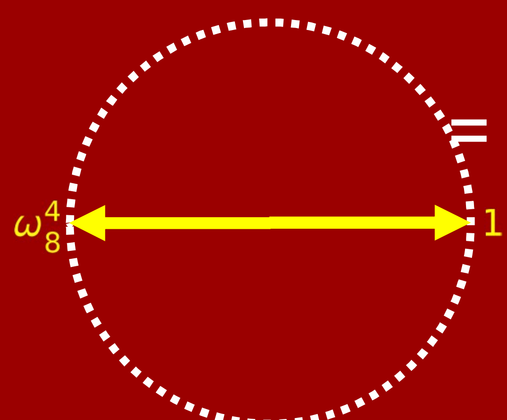
$$\text{IDFT}_8 \qquad\qquad\qquad\qquad \text{DFT}_8$$

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

$$\frac{1+\omega^4+1+\omega^4+1+\omega^4+1+\omega^4+}{8}$$

average

is 0



$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

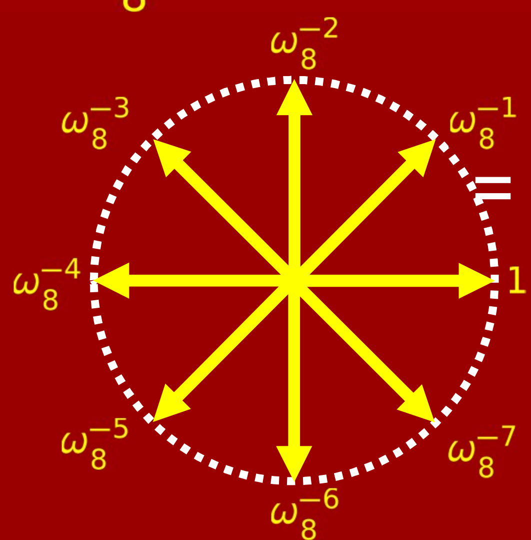
$$\text{IDFT}_8 \qquad\qquad \text{DFT}_8$$

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1 + \omega^{-1} + \omega^{-2} + \omega^{-3} + \omega^{-4} + \omega^{-5} + \omega^{-6} + \omega^{-7}}{8}$$

average is 0



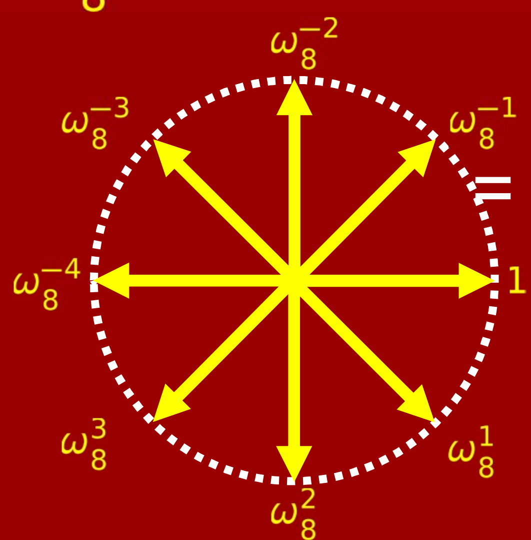
IDFT$_8$

DFT$_8$

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

$$\frac{1 + \omega^1 + \omega^2 + \omega^3 + \omega^{-4} + \omega^{-3} + \omega^{-2} + \omega^{-1}}{8}$$

average is 0



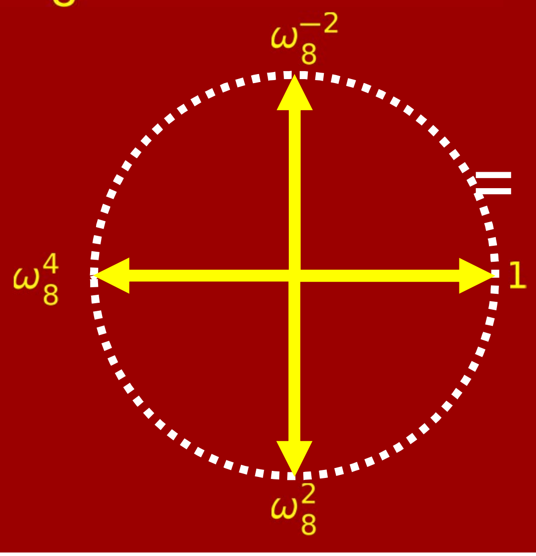
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

IDFT_8 $\qquad\qquad$ DFT_8

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

$$\frac{1+\omega^2+\omega^4+\omega^{-2}+1+\omega^2+\omega^{-4}+\omega^{-2}}{8}$$

average is 0



$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

IDFT$_8$ DFT$_8$

$$\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \omega^{-4} & \omega^{-5} & \omega^{-6} & \omega^{-7} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-1} & \omega^{-4} & \omega^{-7} & \omega^{-2} & \omega^{-5} \\ 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} & 1 & \omega^{-4} \\ 1 & \omega^{-5} & \omega^{-2} & \omega^{-7} & \omega^{-4} & \omega^{-1} & \omega^{-6} & \omega^{-3} \\ 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} & 1 & \omega^{-6} & \omega^{-4} & \omega^{-2} \\ 1 & \omega^{-7} & \omega^{-6} & \omega^{-5} & \omega^{-4} & \omega^{-3} & \omega^{-2} & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{1} & \omega^{2} & \omega^{3} & \omega^{4} & \omega^{5} & \omega^{6} & \omega^{7} \\ 1 & \omega^{2} & \omega^{4} & \omega^{6} & 1 & \omega^{2} & \omega^{4} & \omega^{6} \\ 1 & \omega^{3} & \omega^{6} & \omega^{1} & \omega^{4} & \omega^{7} & \omega^{2} & \omega^{5} \\ 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} & 1 & \omega^{4} \\ 1 & \omega^{5} & \omega^{2} & \omega^{7} & \omega^{4} & \omega^{1} & \omega^{6} & \omega^{3} \\ 1 & \omega^{6} & \omega^{4} & \omega^{2} & 1 & \omega^{6} & \omega^{4} & \omega^{2} \\ 1 & \omega^{7} & \omega^{6} & \omega^{5} & \omega^{4} & \omega^{3} & \omega^{2} & \omega \end{bmatrix}$$

Well, looks pretty true.

See the notes for a

Formal proof.

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Last piece of the puzzle: FFT

$$\mathrm{DFT}_N \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{N-2} \\ a_{N-1} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_{N-2} \\ b_{N-1} \end{bmatrix}$$

Computing this in O(N log N) ops

$$\mathrm{DFT}_N \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{N-2} \\ a_{N-1} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_{N-2} \\ b_{N-1} \end{bmatrix}$$

Claim: DFT_N reduces to 2 applications of $\mathrm{DFT}_{N/2}$, plus $O(N)$ additional operations.

$\Rightarrow T(N) = 2T(N/2) + O(N)$ $\Rightarrow T(N) = O(N \log N)$

Claim: DFT_8 reduces to 2 applications of DFT_4, plus "O(8)" additional operations.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix}$$

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \\ 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$

Claim: DFT_8 reduces to 2 applications of DFT_4,
plus "O(8)" additional operations.

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \\ 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$

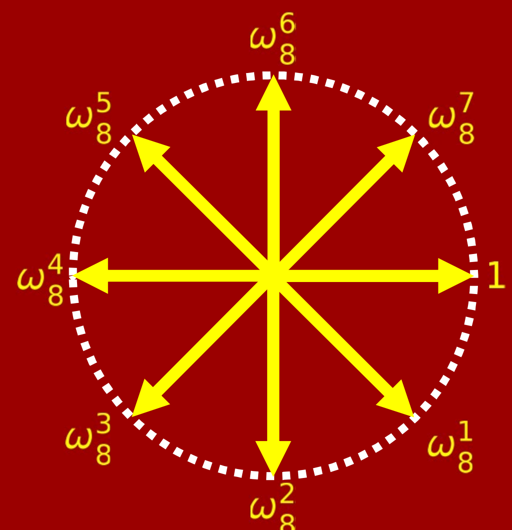
Claim: DFT_8 reduces to 2 applications of DFT_4, plus "O(8)" additional operations.

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \\ 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$

Claim: DFT_8 reduces to 2 applications of DFT_4, plus "O(8)" additional operations.

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \\ 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix} \quad \begin{matrix} = a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix} \\ = \left(\text{ditto} \right) \end{matrix}$$

$$+ a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$



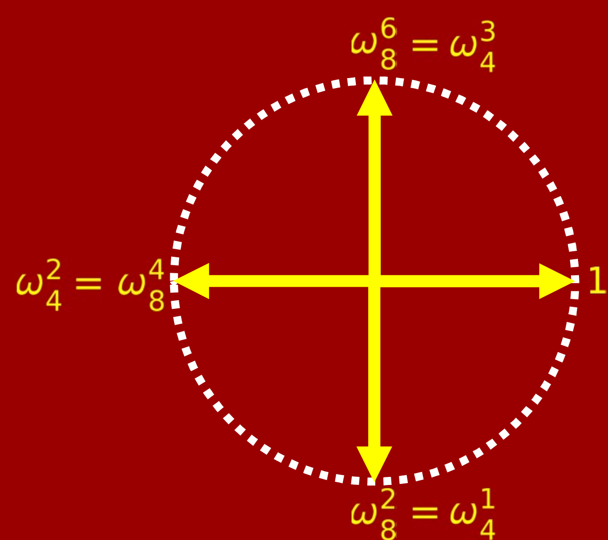
Claim: DFT_8 reduces to 2 applications of DFT_4,
plus “O(8)” additional operations.

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \\ 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix}$$

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix}$$

$$= \left(\text{ditto} \right)$$

$$+ a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$



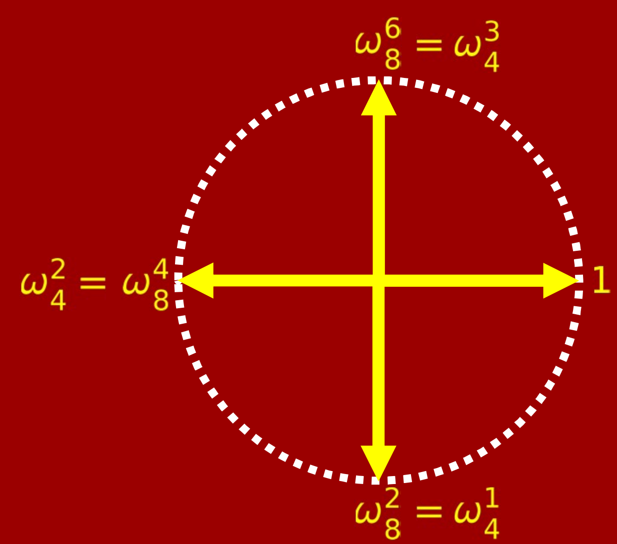
Claim: DFT$_8$ reduces to 2 applications of DFT$_4$, plus "O(8)" additional operations.

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \\ 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix}$$

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega_4^1 \\ \omega_4^2 \\ \omega_4^3 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega_4^2 \\ 1 \\ \omega_4^2 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega_4^3 \\ \omega_4^2 \\ \omega_4^1 \end{bmatrix}$$

$$= \left\{ \text{ditto} \right\}$$

$$+ a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$



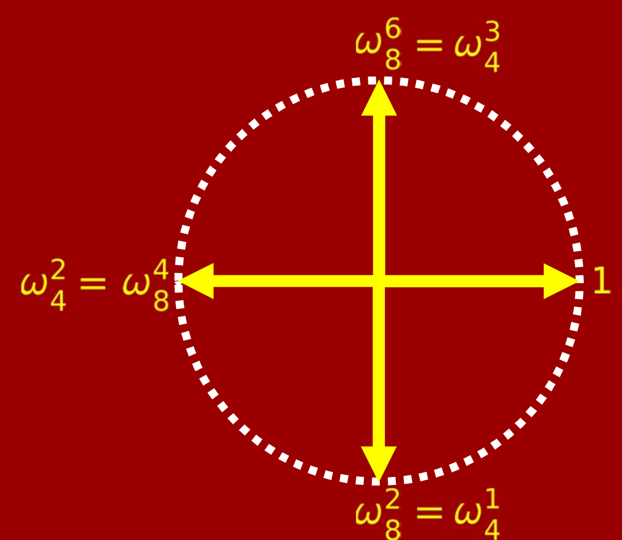
Claim: DFT_8 reduces to 2 applications of DFT_4,
plus "O(8)" additional operations.

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \\ 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_4^1 & \omega_4^2 & \omega_4^3 \\ 1 & \omega_4^2 & 1 & \omega_4^2 \\ 1 & \omega_4^3 & \omega_4^2 & \omega_4^1 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_2 \\ a_4 \\ a_6 \end{bmatrix}$$

$$= \left(\text{ditto} \right)$$

$$+ a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$



Claim: DFT_8 reduces to 2 applications of DFT_4, plus "O(8)" additional operations.

$$= a_0 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \\ 1 \\ \omega^2 \\ \omega^4 \\ \omega^6 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \\ 1 \\ \omega^4 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \\ 1 \\ \omega^6 \\ \omega^4 \\ \omega^2 \end{bmatrix}$$

$$= DFT_4 \cdot \begin{bmatrix} a_0 \\ a_2 \\ a_4 \\ a_6 \end{bmatrix}$$

$$= \left(\text{ditto} \right)$$

$$+ a_1 \cdot \begin{bmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1 \\ \omega^3 \\ \omega^6 \\ \omega \\ \omega^4 \\ \omega^7 \\ \omega^2 \\ \omega^5 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1 \\ \omega^5 \\ \omega^2 \\ \omega^7 \\ \omega^4 \\ \omega \\ \omega^6 \\ \omega^3 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1 \\ \omega^7 \\ \omega^6 \\ \omega^5 \\ \omega^4 \\ \omega^3 \\ \omega^2 \\ \omega \end{bmatrix}$$

Claim: DFT_8 reduces to 2 applications of DFT_4, plus "O(8)" additional operations.

$$P(x) = P^0(x^2) + x\,P^1(x^2)$$

$$= a_0 \cdot \begin{bmatrix} 1\\1\\1\\1\\1\\1\\1\\1 \end{bmatrix} + a_2 \cdot \begin{bmatrix} 1\\ \omega^2\\ \omega^4\\ \omega^6\\ 1\\ \omega^2\\ \omega^4\\ \omega^6 \end{bmatrix} + a_4 \cdot \begin{bmatrix} 1\\ \omega^4\\ 1\\ \omega^4\\ 1\\ \omega^4\\ 1\\ \omega^4 \end{bmatrix} + a_6 \cdot \begin{bmatrix} 1\\ \omega^6\\ \omega^4\\ \omega^2\\ 1\\ \omega^6\\ \omega^4\\ \omega^2 \end{bmatrix}$$

Computable with 1 application of DFT_4 to (a_0,a_2,a_4,a_6), and some copying.

$$+ a_1 \cdot \begin{bmatrix} 1\\ \omega\\ \omega^2\\ \omega^3\\ \omega^4\\ \omega^5\\ \omega^6\\ \omega^7 \end{bmatrix} + a_3 \cdot \begin{bmatrix} 1\\ \omega^3\\ \omega^6\\ \omega\\ \omega^4\\ \omega^7\\ \omega^2\\ \omega^5 \end{bmatrix} + a_5 \cdot \begin{bmatrix} 1\\ \omega^5\\ \omega^2\\ \omega^7\\ \omega^4\\ \omega\\ \omega^6\\ \omega^3 \end{bmatrix} + a_7 \cdot \begin{bmatrix} 1\\ \omega^7\\ \omega^6\\ \omega^5\\ \omega^4\\ \omega^3\\ \omega^2\\ \omega \end{bmatrix}$$

Now to get this, apply the above to (a_1,a_3,a_5,a_7), and then multiply the j^{th} row by ω^j, for $0 \le j < 7$.

Total: 2 applications of DFT_4, plus "O(8)" more operations.

Summary

- Can multiply two *polynomials* of degree < N in O(N log N) time.
- DFT_N reduces Coefficients Representation to Values Representation over roots of unity.
- FFT_N computes DFT_N (and inverse) in O(N log N) time.
- DFT_N has many uses in CS & Engineering.