Lecture 13: Linear Programming I

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Outline

- Definition of linear programming and examples
- A linear program to solve max flow and min-cost max flow
- A linear program to solve minimax-optimal strategies in games
- Algorithms for linear programming

Example

- There are 168 hours in a week. Want to allocate our time between
  - studying (S)
  - going to parties (P)
  - everything else (E)
- To survive: $E \geq 56$
- For sanity: $P + E \geq 70$
- To pass courses: $S \geq 60$
- If party a lot, need to study or eat more: $2S + E - 3P \geq 150$
- Is there a *feasible* solution? Yes, S = 80, P = 20, E = 68
- Happiness is 2P + E. Find a feasible solution maximizing this *objective function*

Linear Program

- This is called a *linear program (LP)*
- All constraints are linear in our variables
- Objective function is linear
- Don't allow $S \cdot E \geq 100$, that's a polynomial program. Much harder.

Formal Definition

- Given:
  - n variables $x_1, \ldots, x_n$
  - m linear inequalities in these variables
    - E.g., $3x_1 + 4x_2 \leq 6$, $0 \leq x_1$, $x_1 \leq 3$

- Goal:
  - Find values for the x_i's that satisfy constraints and maximize objective
  - In the feasibility problem just satisfy the constraints

  - What would happen if we allowed strict inequalities $x_1 < 3$?
  - max x_1

Time Allocation Problem

- Variables: S, P, E
- Objective: Maximize 2P + E subject to
- Constraints: S + P + E = 168

$$E \geq 56$$
$$S \geq 60$$
$$2S + E - 3P \geq 150$$
$$P + E \geq 70$$
$$P \geq 0$$

Operations Research Problem

	labor	materials	pollution
plant 1	2	3	15
plant 2	3	4	10
plant 3	4	5	9
plant 4	5	6	7

- Required to make at least 400 cars at plant 3
- Have 3300 hours of labor and 4000 units of material
- At most 12000 units of pollution
- Maximize number of cars made

What are the variables?

x_1, x_2, x_3, x_4 denote the number of cars at plant i

What's our objective?

maximize $x_1 + x_2 + x_3 + x_4$

	labor	materials	pollution
plant 1	2	3	15
plant 2	3	4	10
plant 3	4	5	9
plant 4	5	6	7

Make at least 400 cars at plant 3
3300 hours of labor and 4000 units of material
At most 12000 units of pollution
Maximize number of cars made

Note: linear programming does not give an integral solution (NP-hard)

What are the variables?

x_1, x_2, x_3, x_4 denote the number of cars at plant i

What's our objective?

maximize $x_1 + x_2 + x_3 + x_4$

Constraints: $x_i \geq 0$ for all i

$$x_3 \geq 400$$
$$2x_1 + 3x_2 + 4x_3 + 5x_4 \leq 3300$$
$$3x_1 + 4x_2 + 5x_3 + 6x_4 \leq 4000$$
$$15x_1 + 10x_2 + 9x_3 + 7x_4 \leq 12000$$

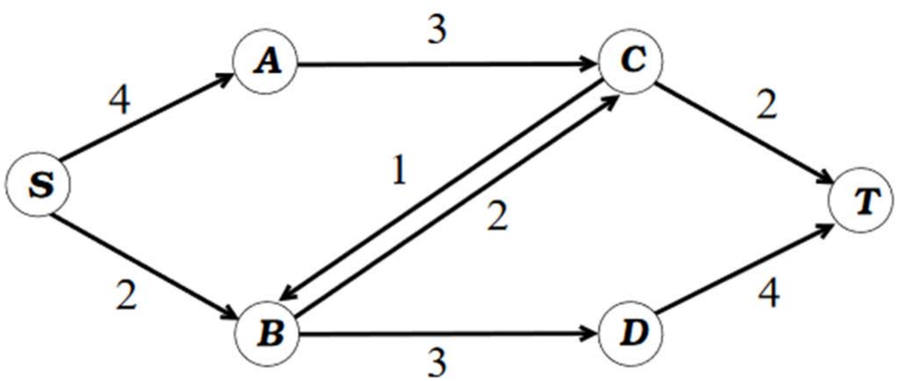
Modeling Network Flow

Variables: f_{uv} for each edge (u,v), representing positive flow

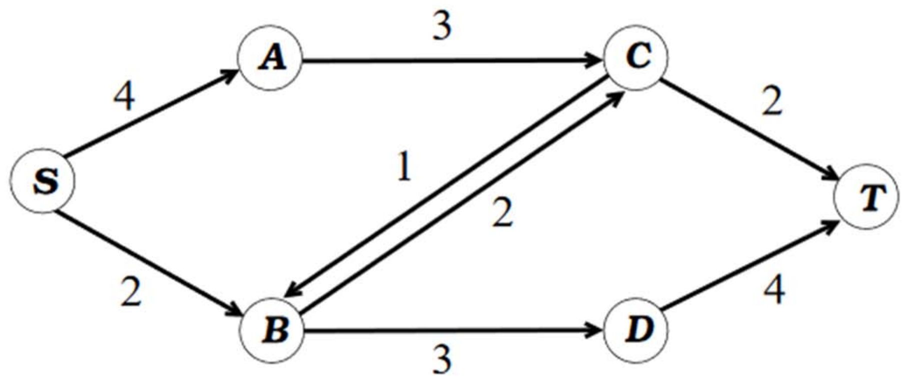
Objective: maximize $\sum_u f_{ut}$

Constraints: For all edges (u,v) $0 \leq f_{uv} \leq c(u,v)$ (capacity constraints)

For all v $\notin \{s,t\}, \sum_u f_{uv} = \sum_u f_{vu}$ (flow conservation)



Modeling Network Flow



In this case, our LP is: maximize $f_{ct} + f_{dt}$ subject to the constraints:

$$0 \le f_{sa} \le 4,\ 0 \le f_{ac} \le 3, \text{ etc.}$$

$$f_{sa} = f_{ac},\ f_{sb} + f_{cb} = f_{bc} + f_{bd},\ f_{ac} + f_{bc} = f_{cb} + f_{ct},\ f_{bd} = f_{dt}.$$

Min Cost Max Flow

- Edge (u,v) has a capacity c(u,v) and a cost w(u,v)
- Find a max s-t flow of least total cost, where the cost of flow f is

$$\sum_{(u,v)\in E} w(u,v) f_{uv}$$

- *How to solve this?*
- Solution 1: Solve for a maximum flow f

  Add a constraint that flow must equal the flow of f

  Minimize $\sum_{(u,v)\in E} w(u,v) f_{uv}$ also subject to original constraints
- Solution 2: Add an edge (t,s) of infinite capacity and very negative cost

  Minimizing cost automatically maximizes flow

Zero Sum Games

Row payoffs:

20	−10	5
5	10	−10
−5	0	10

- Given a zero-sum game with n rows and n columns, compute a minimax optimal strategy for row player

- *What are the variables?*
  - Probabilities $p_1, \ldots, p_n$ on our actions
  - Linear constraints: $\sum_{i=1,\ldots,n} p_i = 1$ and $p_i \geq 0$ for all i
  - Maximize the minimum expected payoff, over all column pure strategies

- *How to maximize a minimum with a linear program?*
- Create new "dummy variable" v to represent minimum

Zero Sum Games

- $\mathrm{R}_{\mathrm{i,j}}$ represents payoff to row player with row player action i and column player action j

- Variables: $\mathrm{p}_1, \ldots, \mathrm{p}_\mathrm{n}$ and v

- Objective: maximize v

- Constraints:
  - $\mathrm{p}_\mathrm{i} \geq 0$ for all i, and $\sum_\mathrm{i} \mathrm{p}_\mathrm{i} = 1$
  - For all columns j, $\sum_\mathrm{i} \mathrm{p}_\mathrm{i} \mathrm{R}_{\mathrm{ij}} \geq \mathrm{v}$

Linear Programs in Standard Form?

- Many different ways to write the same LP
- Use vector notation, so $c^T x = \sum_{i=1,\ldots,d} c_i x_i$ if there are d variables
- Any LP can be written in the following form:
- Max $c^T x$

  Subject to $Ax \leq b$

  $x \geq 0$

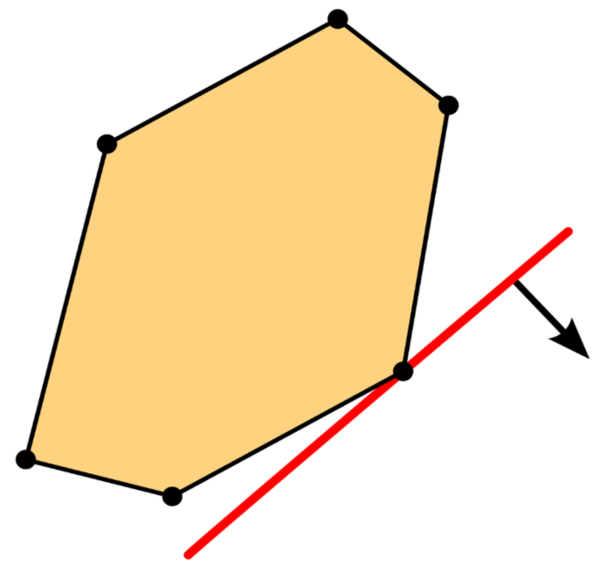
How to handle equality constraints $d^T x$ = e?

How to convert min $c^T x$ to a maximization?

How to handle an unconstrained variable x_i which could be positive or negative?

Substitute $x_i = y_i - z_i$, $y_i \geq 0, z_i \geq 0$, everywhere

Facts about Linear Programs



- Consider the LP
- Max $c^T x$

  Subject to $Ax \leq b$

  $x \geq 0$
- Think of maximizing $c^T x$ over the set $Ax \leq b, x \geq 0$
- What does the set $Ax \leq b, x \geq 0$ look like?
  - Each row is a *halfspace,* cutting R^d into two pieces by a hyperplane
  - The intersection of halfspaces could be empty
    - Then the LP is *infeasible*
  - Could be unbounded
  - Could be bounded and then we call it the *feasible region*
- Maximizing $c^T x$ moves the hyperplane with normal vector c until it is tangent to the feasible region

Convexity Properties

- Feasible region $Ax \leq b, x \geq 0$ is convex
  - If p and q are in the feasible region, then so is the line segment joining p and q. Why?

- Proof by pictures, e.g., convex polygon in two dimensions
- Formally, since $Ap \leq b$ and $Aq \leq b$, for any $\lambda \in [0,1]$,
  - $\lambda Ap \leq \lambda b$ and $(1-\lambda)Aq \leq (1-\lambda)b$
  - So $A(\lambda p + (1-\lambda)q) \leq b$
  - Also $\lambda p \geq 0$ and $(1-\lambda)q \geq 0$ since $p \geq 0$ and $q \geq 0$

- More generally, intersections of convex sets are convex
- Max $c^T x$ occurs at a vertex. Can we just enumerate all vertices?

Algorithms for Linear Programming

- Simplex Algorithm
  - Practical, but exponential time in the worst-case

- Ellipsoid Algorithm
  - First polynomial time algorithm, but slow in practice

- Karmarkar's Algorithm (interior point)
  - Polynomial time algorithm and competitive in practice

- Software: LINDO, CPLEX, Solver (in Excel)

Time Allocation Problem

- Variables: S, P, E
- Objective: Maximize 2P + E subject to
- Constraints: S + P + E = 168

$$E \geq 56$$
$$S \geq 60$$
$$2S + E - 3P \geq 150$$
$$P + E \geq 70$$
$$P \geq 0$$

Substitute S = 168-P-E, so two variables P and E, want to maximize 2P+E.

Intuition for Linear Programming

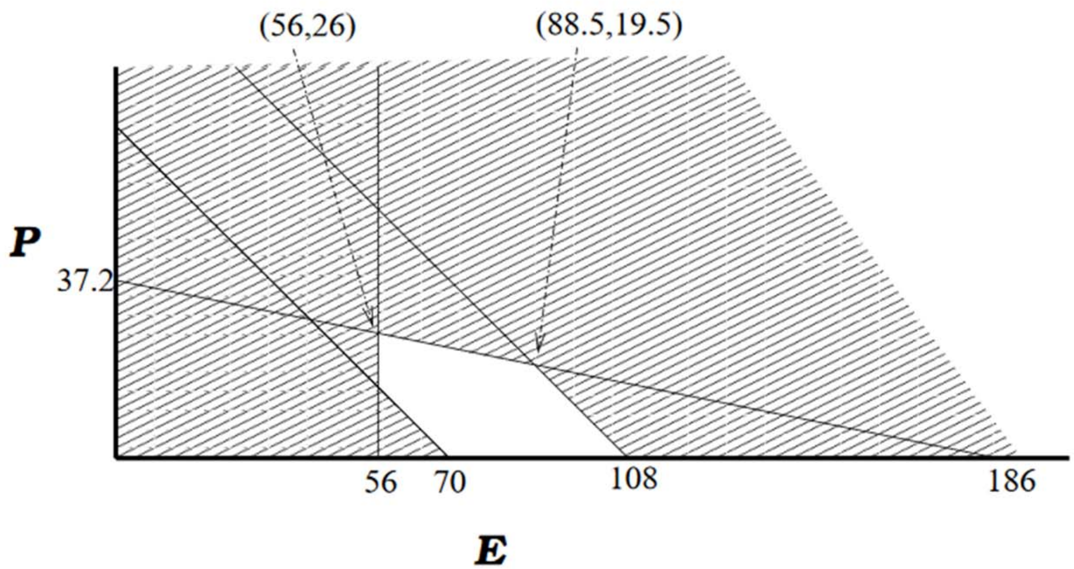
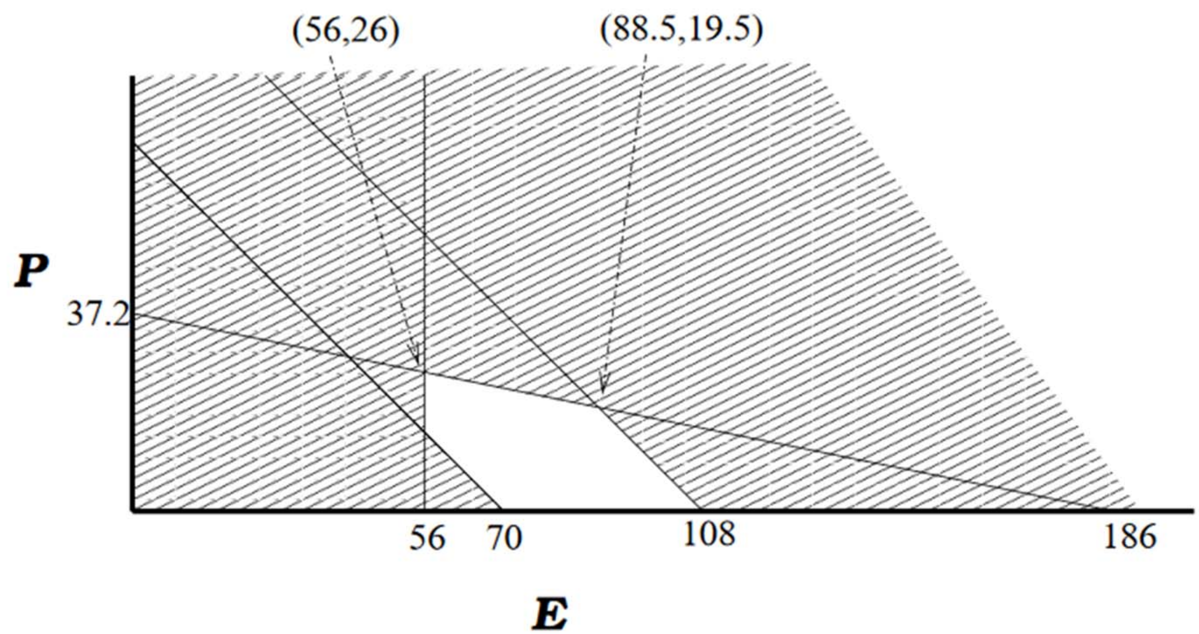


Figure 13.1: Feasible region for our time-planning problem. The constraints are: $E \geq 56$; $P + E \geq 70$; $P \geq 0$; $S \geq 60$ which means $168 - P - E \geq 60$ or $P + E \leq 108$; and finally $2S - 3P + E \geq 150$ which means $2(168 - P - E) - 3P + E \geq 150$ or $5P + E \leq 186$.

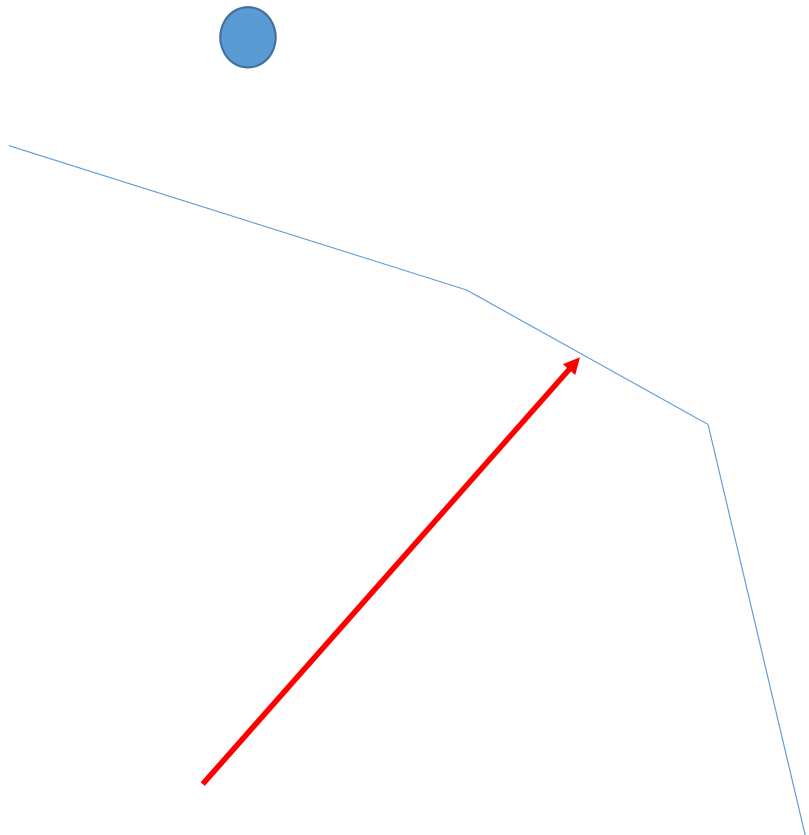
Maximizing P occurs at (56,26). Maximizing 2P+E occurs at (88.5, 19.5)

Simplex Algorithm



Start at vertex of the feasible region (polyhedron in high dimensions)
Look at cost of objective function at each neighbor
Move to neighbor of minimum cost
Always make progress, but could take exponential time (in high dimensions)

Simplex Algorithm



Get stuck in local maximum?

No, since feasible set is convex

Other Annoyances I

- How to start at a vertex of the feasible region?
- $Ax \leq b$
  $x \geq 0$
- What if it's not even feasible?
- Introduce "slack" variable s. Consider:
- $\min s$

  subject to $Ax \leq b + s \cdot 1^m$

  $$x \geq 0, s \geq 0, s \leq \max_i -b_i$$
- Feasible. Can run simplex starting at x = 0^n and s = $\max_i -b_i$
- If original LP is feasible, minimum achieved when s = 0, and x that is output is a vertex in the feasible region of original LP

Other Annoyances II

- What if the feasible region is unbounded?
  - Ok, as long as objective function is bounded

- What if objective function is unbounded?
  - Output ∞, how to detect this?

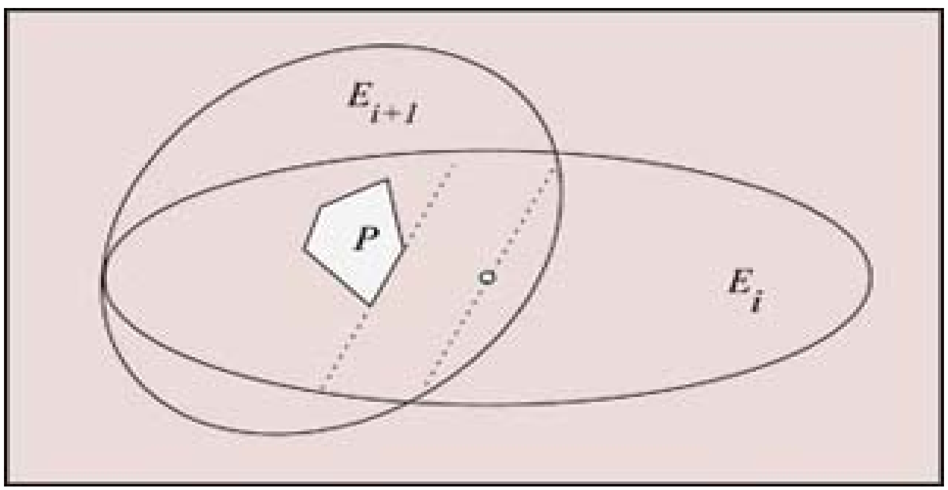
- Many ways
  - see one based on duality in a few lectures
  - include constraints $x_i \leq M$ for all i, for a very large value M
  - can efficiently find M to ensure if solution is finite, still find the optimum

Ellipsoid Algorithm

Solves feasibility problem

Replace objective function with constraint, do binary search

Replace “minimize $x_1 + x_2$" with $x_1 + x_2 \leq \lambda$



Can handle exponential number of constraints if there’s a separation oracle

Karmarkar's Algorithm

- Works with feasible points but doesn't go corner to corner
- Moves in interior of the feasible region – "interior point method"