

In this lecture we describe the *union-find* problem. This is a problem that captures a key task one needs to solve in order to efficiently implement Kruskal's minimum-spanning-tree algorithm. We then give two data structures for union-find with good amortized running time.

1 Motivation

To motivate the union-find problem, let's recall Kruskal's Algorithm for finding a minimum spanning tree (MST) in an undirected graph (see also Section 5). Remember that an MST is a tree that includes (i.e., spans) all the vertices and out of all such trees has the least total cost.

Kruskal's Algorithm (recap):

Sort the edges in the given graph G by length and examine them from shortest to longest.
Put each edge into the current forest if it doesn't form a cycle with the edges chosen so far.

We argue correctness in Section 5.2. Today, our concern is running time. The initial step takes time $O(|E| \log |E|)$ to sort. Then, for each edge, we need to test if it connects two different components. If it does, we will insert the edge, merging the two components into one; if it doesn't (the two endpoints are in the same component), then we will skip this edge and go on to the next edge. So, to do this efficiently we need a data structure that can support the basic operations of (a) determining if two nodes are in the same component, and (b) merging two components together. This is the *union-find* problem.

2 The Union-Find Problem

The general setting for the union-find problem is that we are maintaining a collection of disjoint sets $\{S_1, S_2, \dots, S_k\}$ over some universe, with the following operations:

MakeSet(x): create the set $\{x\}$.

Union(x, y): replace the set x is in (let's call it S) and the set y is in (let's call it S') with the single set $S \cup S'$.

Find(x): return the unique ID for the set containing x (this is just some representative element of this set).

Given these operations, we can implement Kruskal's algorithm as follows. The sets S_i will be the sets of vertices in the different trees in our forest. We begin with **MakeSet**(v) for all vertices v (every vertex is in its own tree). When we consider some edge (v, w) in the algorithm, we just test whether **Find**(v) equals **Find**(w). If they are equal, it means that v and w are already in the same tree so we skip over the edge. If they are not equal, we insert the edge into our forest and perform a **Union**(v, w) operation. All together we will do $|V|$ **MakeSet** operations, $|V| - 1$ **Unions**, and $2|E|$ **Find** operations.

Notation and Preliminaries: in the discussion below, it will be convenient to define

- n as the number of MakeSet operations and
- m as the total number of operations

(this matches the number of vertices and edges in the graph up to constant factors, and so is a reasonable use of n and m). Also, it is easiest to think conceptually of these data structures as adding fields to the items themselves, so there is never an issue of “how do I locate a given element v in the structure?”.

3 Data Structure 1 (list-based)

Our first data structure is a simple one with a very cute analysis. The total cost for the operations will be $O(m + n \log n)$.

In this data structure, the sets will be just represented as linked lists: each element has a pointer to the next element in its list. However, we will augment the list so that each element also has a pointer directly to head of its list (denoted by $x \rightarrow \text{head}$). The head of the list is the representative element. We can now implement the operations as follows:

MakeSet(x): just set $x \rightarrow \text{head} = x$. This takes constant time.

Find(x): just return $x \rightarrow \text{head}$. Also takes constant time.

Union(x, y): To perform a union operation we merge the two lists together, and reset the head pointers on one of the lists to point to the head of the other.

Let A be the list containing x and B be the list containing y , with lengths L_A and L_B respectively. Then we can do this in time $O(L_A + L_B)$ by appending B onto the end of A as follows. We first walk down A to the end, and set the final **next** pointer to point to $y \rightarrow \text{head}$. This takes time $O(L_A)$. Next we go to $y \rightarrow \text{head}$ and walk down B , resetting head pointers of elements in B to point to $x \rightarrow \text{head}$. This takes time $O(L_B)$.

Can we reduce this to just $O(L_B)$? Yes. Instead of appending B onto the end of A , we can just splice B into the middle of A , at x . I.e., let $z = x \rightarrow \text{next}$, set $x \rightarrow \text{next} = y \rightarrow \text{head}$, then walk down B as above, and finally set the final **next** pointer of B to z .

Can we reduce this to $O(\min(L_A, L_B))$? Yes. Just store the length of each list in the head. Then compare and insert the shorter list into the middle of the longer one. Then update the length count to $L_A + L_B$.

We now prove this simple data structure has the running time we wanted.

Theorem 1 *The above algorithm has total running time $O(m + n \log n)$.*

Proof: The Find and MakeSet operations are constant time so they are covered by the $O(m)$ term. Each Union operation has cost proportional to the length of the list whose head pointers get updated. So, we need to find some way of analyzing the total cost of the Union operations.

Here is the key idea: we can pay for the union operation by charging $O(1)$ to each element whose head pointer is updated. So, all we need to do is sum up the costs charged to all the elements over the entire course of the algorithm. Let's do this by looking from the point of view of some lowly

element x . Over time, how many times does x get walked on and have its head pointer updated? The answer is that its head pointer is updated at most $\log n$ times. The reason is that we only update head pointers on the *smaller* of the two lists being joined, so every time x gets updated, the size of the list it is in at least doubles, and this can happen at most $\log n$ times. So, we were able to pay for unions by charging the elements whose head pointers are updated, and no element gets charged more than $O(\log n)$ total, so the total cost for unions is $O(n \log n)$, or $O(m + n \log n)$ for all the operations together. ■

Recall that this is already low-order compared to the $O(m \log m)$ sorting time for Kruskal's algorithm. So we could use this to get $O(m \log m)$ overall runtime for Kruskal.

Exercise 1: After doing n `makeSet` operations, give an sequence of $n - 1$ `union` operations that causes the above data structure to take $\Omega(n \log n)$ time.

4 Data Structure 2 (tree-based)

But even though the running time of the list-based data structure is pretty fast, let's think of ways we could make it even faster. How fast *can* we make a union-find data structure?

One idea is that instead of updating all the head pointers in list B (or whichever was shorter) when we perform a Union, we could do this in a lazy way, just pointing the head of B to the head of A and then waiting until we actually perform a find operation on some item x before updating its pointer. This will decrease the cost of the Union operations but will increase the cost of Find operations because we may have to take multiple hops. Notice that by doing this we no longer need the downward pointers: what we have in general is a collection of trees, with all links pointing *up*. Another idea is that rather than deciding which of the two heads (or roots) should be the new one based on the *size* of their sets, perhaps there is some other quantity that would give us better performance. In particular, it turns out we can do better by setting the new root based on which tree has larger *rank*, which we will define in a minute.

We will prove that by implementing the two optimizations described above (lazy updates and union-by-rank), the total cost is bounded above by $O(m \lg^* n)$, where recall that $\lg^* n$ is the number of times you need to take \log_2 until you get down to 1. For instance,

$$\lg^*(2^{65536}) = 1 + \lg^*(65536) = 2 + \lg^*(16) = 3 + \lg^*(4) = 4 + \lg^*(2) = 5.$$

So, in practical settings, $\lg^* n$ is never bigger than 5. However, it still is not a constant, since $\lg^* n \rightarrow \infty$ as $n \rightarrow \infty$.¹

We now describe the procedure more specifically. Each element (node) will have two fields: a *parent* pointer that points to its parent in its tree (or itself if it is the root) and a rank, which is an integer used to determine which node becomes the new root in a Union operation. The operations are:

MakeSet(x): set x 's rank to 0 and its parent pointer to itself. This takes constant time.

Find(x): starting from x , follow the parent pointers until you reach the root, updating x and all the nodes we pass over to point to the root. This is called *path compression*.

The running time for Find(x) is proportional to (original) distance of x to its root.

¹As an aside, the running time of the union-find data structure is even better: $O(m\alpha(m, n))$ where α is the inverse-Ackermann function which grows even more slowly than \lg^* . But the $\lg^* n$ bound is subtle enough to prove — let's not go completely overboard!

Union(x, y): Let $\text{Union}(x, y) = \text{Link}(\text{Find}(x), \text{Find}(y))$, where Link is an internal operation that is not part of the public interface, and is defined as follows.

Link(r_1, r_2): The invariant is that the two arguments to Link are always roots. If the one of the roots (say r_2) has strictly larger rank than the other, then r_2 becomes the new root, and r_1 's parent pointer is changed to point to r_2 . If the two roots have *equal* rank, then one of them (arbitrarily, say r_1) is picked to be the new root, r_2 's parent pointer points to r_1 , and r_1 's rank is increased by 1. This procedure is called *union by rank*.

Properties of ranks: To help us understand this procedure, let's first develop some properties of ranks.

- (A) The rank of a node is the same as what the height of its subtree would be if we didn't do path compression. This is easy to see: if you take two trees of *different* heights and join them by making the root of the shorter tree into a child of the root of the taller tree, the heights do not change, but if the trees were the *same* height, then the final tree will have its height increase by 1.
- (B) If x is not a root, then $\text{rank}(x)$ is strictly less than the rank of x 's parent. We can see this by induction: the Union operation maintains this property, and the Find operation only increases the difference between the ranks of nodes and their parents.
- (C) This means that when we do path compression, if x 's parent changes, then the rank of x 's new parent is *strictly more* than the rank of x 's old parent.
- (D) The rank of a node x can only change if x is a root. Furthermore, once a node becomes a non-root, it is never a root again. These are immediate from the algorithm.
- (E) There are at most $n/2^r$ nodes of rank $\geq r$.

Indeed, when a node x first reaches rank r (due to a **link**), it must be a root (ppty D), its tree must have at least 2^r nodes (this is easy to see by induction), all these nodes (except for itself) have rank $< r$ (ppty B), and their ranks are never going to change (ppty D). Define S_x to be these nodes in its subtree *at this moment*. Note that these nodes did not have an ancestor node with rank r before the link, but now they do. Moreover, path compression will maintain the property that they have an ancestor of rank $\geq r$ (ppty C).

Now consider some y that reaches rank r later, and let S_y be the (at least 2^r) nodes in its tree at this moment. Since these nodes did not have an ancestor with rank $\geq r$ before this **link** (for the same reason), and those in S_x already did, S_y cannot have any intersection with S_x .

So for any two nodes x and y of rank $\geq r$, the sets S_x and S_y are disjoint, and each has size $\geq 2^r$. Since there are n nodes total, this implies there can be at most $n/2^r$ nodes of rank $\geq r$.

Exercise 2: If there are n elements, show that every element has rank $\leq \log_2 n$.

Using the facts above, we can warm up by showing that the (worst-case) cost for each find operation is $O(\log n)$. Here are two ways to do it:

Proof 1: The rank of a node's parent is strictly greater than its parent. So ranks strictly increase as we traverse a path towards the root. Since the maximum rank of any node is $\log_2 n$, the length of any path followed in a find operation is at most $\log_2 n$. Hence m find operations cost at most $O(m \log n)$ cost.

Proof 2: We claim that each node helps others $O(\log n)$ times. Indeed, when you do a $\text{find}(x)$, you traverse a path from x to its root. Say this path is $x = v_0, v_1, v_2, \dots, v_{k-2}, v_{k-1}, v_k = \text{root}$. The length of the path is k . Who pays for this?

Suppose we say that all nodes on the path, except the root and its immediate child, all help pay for this find on x . This way we can get $k - 2$ dollars. The find operation can itself pay the remaining 2 dollars. So it remains to see how much money some generic node (say v_i) pays to help others? Well, each time v_i pays money its parent changes from v_{i+1} to v_k , due to path compression. And the rank of this new parent of v_i is strictly more than the rank of the old parent. The final rank of its parent cannot be more than $\log_2 n$. So node v_i will pay at most $\log_2 n$ times to help others. This means the total cost of m operations is $O(m + n \log n)$, since each operation can pay $O(1)$ to take care of small expenses (like the last two nodes on the path, and other small things), and each node pays $O(\log n)$ to help other nodes.

4.1 The $O(m \lg^* n)$ Proof

We're now ready to prove the following theorem.²

Theorem 2 *The above tree-based algorithm has total running time $O(m \lg^* n)$.*

Proof: Let's begin with the easy parts. First of all, the Union does two Find operations plus a constant amount of extra work. So, we only need to worry about the time for the (at most $2m$) Find operations.

Second, we can count the cost of a Find operation by charging \$1 for each parent pointer examined. So, when we do a $\text{Find}(x)$, if x was a root then pay \$1 (just a constant, so that's ok). If x was a child of a root we pay \$2 (also just a constant, so that's ok also). If x was lower, then the *very rough* idea is that (except for the last \$2) every dollar we spend is shrinking the tree because of our path compression, so we'll be able to amortize this cost somehow. For the remaining part of the proof, we will use the properties we figured out above.

Step 1: first let's imagine putting non-root nodes into buckets according to their rank.

- Bucket 0 contains all non-root nodes of rank 0,
- bucket 1 has all of rank 1,
- bucket 2 has ranks 2 through $2^2 - 1$,
- bucket 3 has ranks 2^2 through $2^{2^2} - 1$,
- bucket 4 has ranks 2^{2^2} through $2^{2^{2^2}} - 1$, etc.
- In general, a bucket has ranks r through $2^r - 1$. We'll denote the bucket as $[r, 2^r - 1]$.

In total, we have $O(\lg^* n)$ buckets, and each node belongs to one bucket.

How many nodes have ranks in bucket $[r, 2^r - 1]$? All these nodes have ranks at least r , so by property (E) of ranks above, this is at most $\frac{n}{2^r} = \frac{n}{\text{upper bound of bucket} + 1}$.

Step 2: When we walk up the tree in the $\text{Find}(x)$ operation, some one must pay for it. We use the banker's method to account for it.

When we do $\text{Find}(x)$, we give the queried element x $2 + \lg^* n$ tokens (call these *red* tokens). But this may not be enough, if the path is longer than $2 + \lg^* n$. So we give some more money to each non-root element: each element that is not a root, and lies in bucket $[r, 2^r - 1]$, gets 2^r *green* tokens

²Optional for Spring 2019, we could not cover it in lecture.

to help others. There are at most $n/2^r$ elements in this bucket (ppty E), and each gets 2^r green tokens, so that's n green tokens per bucket, or $O(n \lg^* n)$ green tokens in all.

Now consider the path from x to its root, and some step $u \rightarrow v$ along this path. If the step is such that v is in the *same* bucket as u , then node u (the *helper*) uses its green tokens to pay for this step. But if v is in a higher bucket, then x (the *walker*) pays for it out of its red tokens. The last two steps, when we touch the root, or a child of the root, we'll charge to the walker again, but this is only \$2.

The easy part of this is the amount x pays. We can move up in buckets at most $\lg^* n$ times, since there are only $\lg^* n$ different buckets. So, the total amount the “walker” x pays is $2 + \lg^* n$, at most the amount of red tokens we gave it.

The slightly harder part is: how much does a helper u pay? The argument is careful but simple.

1. When node u pays using its green tokens, it is not a root, so its rank is never going to change, by Property (D).
2. Every time u pays, the rank of its new parent (after path compression) is at least 1 larger than the rank of its previous parent, by property (C).
3. One worry: the rank of u 's parent could conceivably increase $\log n$ times, which to us is a “big” number. Hmm.

But — and this is the crucial idea — once its parent's rank becomes large enough that it is in the next bucket, we never charge node u again as helper. So, the maximum amount any helper node u pays is at most the range of its bucket $[r, 2^r - 1]$, i.e., at most 2^r . Which is the amount of green tokens we gave it.

(Remember that the only elements being charged are non-roots, so once they start getting charged, their rank is fixed so they can't jump to some other bucket and start getting charged there too.)

4. So every operation can be paid for by either a red or green token. The total number of tokens is at most $m(\lg^* n + 2)$ red tokens, plus $n \lg^* n$ green tokens. Since $m \geq n$, this proves the theorem.

Simple, and beautiful. Just as advertised. ■

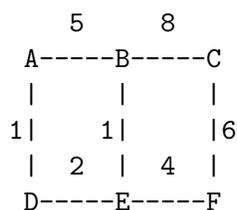
Exercise 3: Show that the charge to the walkers, and hence the total cost of n **makesets** and m **unions** and **finds**, is only $O(m + n \lg^* n)$. This is often better than $O(m \lg^* n)$ when $m \geq n$.

5 Appendix: MST Algorithms

Many of you have seen minimum spanning tree algorithms in previous courses, e.g., in 15-210 you saw Boruvka's algorithm, which is naturally parallel and runs in time $O(m \log n)$. But let us recap the basic definitions, and talk about two other algorithms: Prim's and Kruskal's.

A **spanning tree** of a graph is a tree that touches all the vertices (so, it only makes sense in a connected graph). A **minimum spanning tree** (MST) is a spanning tree whose sum of edge lengths is as short as possible (there may be more than one). We will sometimes call the sum of edge lengths in a tree the *size* of the tree. For instance, imagine you are setting up a communication network among a set of sites and you want to use the least amount of wire possible. *Note:* our definition is only for *undirected* graphs.

What is the MST in the graph below?



5.1 Prim's algorithm

Prim's algorithm is an MST algorithm that works much like Dijkstra's algorithm does for shortest path trees, if you are familiar with that. In fact, it's even simpler (though the correctness proof is a bit trickier).

Prim's Algorithm:

1. Pick some arbitrary start node s . Initialize tree $T = \{s\}$.
2. Repeatedly add the shortest edge incident to T (the shortest edge having one vertex in T and one vertex not in T) until the tree spans all the nodes.

For instance, what does Prim's algorithm do on the above graph? ³

Before proving correctness for the algorithm, we first need a useful fact about spanning trees: if you take any spanning tree and add a new edge to it, this creates a cycle. The reason is that there already was one path between the endpoints (since it's a *spanning* tree), and now there are two. If you then remove any edge in the cycle, you get back a spanning tree (removing one edge from a cycle cannot disconnect a graph).

Theorem 3 *Prim's algorithm correctly finds a minimum spanning tree of the given graph.*

Proof: We will prove correctness by induction. Let G be the given graph. Our inductive hypothesis will be that the tree T constructed so far is consistent with (is a subtree of) some minimum spanning tree M of G . This is certainly true at the start. Now, let e be the edge chosen by the algorithm. We need to argue that the new tree, $T \cup \{e\}$ is also consistent with some minimum spanning tree M' of G . If $e \in M$ then we are done ($M' = M$). Else, we argue as follows.

Consider adding e to M . As noted above, this creates a cycle. Since e has one endpoint in T and one outside T , if we trace around this cycle we must eventually get to an edge e' that goes back in to T . We know $len(e') \geq len(e)$ by definition of the algorithm. So, if we add e to M and remove e' , we get a new tree M' that is no larger than M was and contains $T \cup \{e\}$, maintaining our induction and proving the theorem. ■

Running time: To implement this efficiently, we can store the neighbors of the current tree in a priority-queue (pqueue), with priority-value equal to the length of the shortest edge between that node and the current tree. We add a new node into the tree using a remove-min operation (taking the node of smallest priority-value out of the pqqueue); then, after adding this node, we examine all outgoing edges and for each one that points to a node not in the tree we either (a) add it into

³In case you want to handle disconnected graphs and find a "min-weight spanning forest", just run Prim's on each component!

the pqueue if it is not there already, or (b) perform a “decrease-key” operation if it was in there already but the new edge is shorter. This will give us $O(m \log n)$ running time if we implement the pqueue using a standard heap, or $O(m + n \log n)$ running time if we use something called a Fibonacci heap.

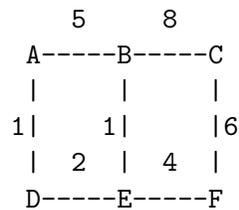
5.2 Kruskal’s algorithm

Here is another algorithm for finding minimum spanning trees called Kruskal’s algorithm. It is also greedy but works in a different way.

Kruskal’s Algorithm:

Sort edges by length and examine them from shortest to longest. Put each edge into the current forest (a forest is just a set of trees) if it doesn’t form a cycle with the edges chosen so far.

E.g., let’s look at how it behaves in the graph below:



Kruskal’s algorithm sorts the edges and then puts them in one at a time so long as they don’t form a cycle. So, first the AD and BE edges will be added, then the DE edge, and then the EF edge. The AB edge will be skipped over because it forms a cycle, and finally the CF edge will be added (at that point you can either notice that you have included $n - 1$ edges and therefore are done, or else keep going, skipping over all the remaining edges one at a time).

Theorem 4 *Kruskal’s algorithm correctly finds a minimum spanning tree of the given graph.*

Proof: We can use a similar argument to the one we used for Prim’s algorithm. Let G be the given graph, and let F be the forest we have constructed so far (initially, F consists of n trees of 1 node each, and at each step two trees get merged until finally F is just a single tree at the end). Assume by induction that there exists an MST M of G that is consistent with F , i.e., all edges in F are also in M ; this is clearly true at the start when F has no edges. Let e be the next edge added by the algorithm. Our goal is to show that there exists an MST M' of G consistent with $F \cup \{e\}$.

If $e \in M$ then we are done ($M' = M$). Else add e into M , creating a cycle. Since the two endpoints of e were in different trees of F , if you follow around the cycle you must eventually traverse some edge $e' \neq e$ whose endpoints are also in two different trees of F (because you eventually have to get back to the node you started from). Now, both e and e' were eligible to be added into F , which by definition of our algorithm means that $len(e) \leq len(e')$. So, adding e and removing e' from M creates a tree M' that is also a MST and contains $F \cup \{e\}$, as desired. ■

Running time: The first step is sorting the edges by length which takes time $O(m \log m)$. Then, for each edge we need to test if it connects two different components. This seems like it should be a real pain: how can we tell if an edge has both endpoints in the same component? This is just the union-find data structure you saw in this lecture! It is so efficient that union/finds will actually will be a low-order cost compared to the sorting step.