Ten Powerful Ideas
and applications

Final Exam Solutions	Steven Rudich	May 9, 1997

Repeat After Me.

This part is to test your ability to study material and repeat it.

1. [2 points]
Give a formula for n choose k.

$$\frac{n!}{k!(n-k)!}$$

3. [3 points]
Give the definition of a tree.

A tree is a connected graph with no cycles.

3. [5 points]
Show that at any party with six people there are either three mutual acquaintances or three mutual strangers. (You may assume that any two people are either acquaintances or strangers.)

This question is equivalent to a problem we studied in Lecture 25: any 2-coloring of the edges of a complete graph on 6 nodes will contain a monochromatic triangle.

Proof: let v be an arbitrary node in the complete graph. v has 5 neighbors; by the pigeonhole principle, at least 3 of them are connected to v by edges of the same color (call it blue). If two of the 3 neighbors are connected by a blue edge, it forms a blue triangle. If not, then the 3 neighbors themselves form a triangle of the opposite color. QED.

Basic Techniques.

This part will test your ability to reason with some of the basic techniques learned in the class.

4. [5 points]
Every node of a ternary tree has 3 children (except for the leaf nodes, which have no children).

If a one-level ternary tree contains 1 node, how many nodes does an n-level ternary tree have? Your solution must be in closed form. I.e., no summation in the exression that gives the solution.

The ith level of the ternary tree contains 3^i nodes, so a tree with levels $0, 1, \ldots, n-1$ has

$$\begin{aligned}\sum_{i=0}^{n-1} 3^i &= \frac{3^n - 1}{3 - 1} \\ &= \frac{3^n - 1}{2}\end{aligned}$$

5. [5 points]
Prove that $1 + 2 + 3 + \cdots + n$ is divisible by n if and only if n is odd.

By Little Gauss' formula, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. This number is divisible by n if and only if $\frac{n+1}{2}$ is an integer, which is true if and only if $n + 1$ is even, which is true if and only if n is odd.

6. [10 points]
$f(0) = 2$, $f(1) = 1$, for $x > 1$, $f(x) = f(x-1) + 2f(x-2)$. Give a closed form solution for f.

Applying the method of linear recurrence relations, we let $f(x) = \lambda^x$ and solve for λ:

$$\lambda^x = \lambda^{x-1} + 2\lambda^{x-2}$$

Dividing through by λ^{x-2} gives

$$\begin{aligned}\lambda^2 &= \lambda + 2 \\ \lambda^2 - \lambda - 2 &= 0 \\ (\lambda + 1)(\lambda - 2) &= 0\end{aligned}$$

This gives two possible solutions, $\lambda_1 = -1$ and $\lambda_2 = 2$. The solution is some linear combination of these two solutions, $f(x) = a(-1)^x + b2^x$. To find a and b, substitute the base cases:

$$\begin{aligned}f(0) &= a + b = 2 \\ f(1) &= 2a - b = 1\end{aligned}$$

Solving gives $a = 1$ and $b = 1$. Thus the closed form solution for f is

$$f(x) = (-1)^x + 2^x$$

A Moment's Thought!

This section tests your ability to think a little bit more insightfully. You must give complete explanations of your answers. **Do seven of the eight problems. Do not do all eight; if you do we will grade seven of our choice.**

7. [10 points]
Give a closed-form solution to

$$\sum_{k=0}^{n} \binom{n}{k} 3^k (-1)^{n-k}$$

Use the binomial theorem,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

With $x = 3$ and $y = -1$, we get

$$2^n$$

8. [10 points]
Let $x_1, x_2, \ldots, x_m$ be any m integers. Prove that there must exist a pair whose difference is divisible by $m-1$.

By the pigeonhole principle, at least two of the m integers must be congruent modulo $m-1$ – so $x_i = x_j \pmod{m-1}$ for some $i \neq j$. Thus $x_i - x_j = 0 \pmod{m-1}$, so $x_i - x_j$ is divisible by $m-1$.

9. [10 points]
What is the expected sum of a randomly chosen subset S of $\{1, 2, 3, \ldots, 20\}$? (In other words, S is equally likely to be any of the 2^{20} subsets. The sum of S is the sum of all its elements. The sum of the empty set is 0. What is the expectation of this sum?)

Define the random variable X_i as follows: $X_i = i$ if i is in the chosen subset; otherwise $X_i = 0$. Thus $S = X_1 + X_2 + \cdots + X_{20}$. By linearity of expectation, $E[S] = E[X_1] + E[X_2] + \cdots + E[X_{20}]$. Since each i is equally likely to be in the set as not, $E[X_i] = \frac{i}{2}$. Thus

$$\begin{aligned} E[S] &= \sum_{i=1}^{20} \frac{i}{2} \\ &= \frac{1}{2} \sum_{i=1}^{20} i \\ &= \frac{1}{2} \frac{20 \cdot 21}{2} \\ &= 105 \end{aligned}$$

10. [10 points]
Count the number of ways to choose distinct subsets A_0, A_1, $A_2, \ldots, A_n$ of $\{ 1, 2, 3, \ldots, n \}$ such that $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n$. ($A \subset B$ means that A is a subset of B.) Your solution should be inclosed form if possible.

Let us count the ways by assigning elements of the set to the *difference sets*: $A_0, A_1 - A_0, A_2 - A_1, \ldots, A_n - A_{n-1}$. We'll also allow some elements to be assigned to a set X (the trash can), meaning that they aren't used.

Because the subsets must be distinct, each of the difference sets $A_1 - A_0, \ldots, A_n - A_{n-1}$ must contain at least one element. Since there are n such difference sets and only n elements, each difference set must contain exactly one element, and A_0 and X (the leftovers) must be empty sets. So the number of ways to choose the subsets is just the number of permutations of n elements,

$$n!$$

11. [10 points]
Count the number of ways to choose subsets A_0, A_1, $A_2, \ldots, A_n$ of $\{ 1, 2, 3, \ldots, n \}$ such that $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n$. The subsets are *not* required to be distinct. Your solution should be in closed form if possible.

Use the same setup as the previous problem: let us count the number of ways to assigning elements to $A_0, A_1 - A_0, A_2 - A_1, \ldots, A_n - A_{n-1}$, and X (the trash can). Now, however, there are no constraints, so each of the n elements can choose any of the $n + 2$ sets, which gives

$$(n + 2)^n$$

12. [10 points]
Consider a dial that is equally likely to point to each of n regions numbered $1, 2, \ldots, n$. When we spin the dial three times what is the probability that the sum of the selected numbers is n?

Let x_1, x_2, x_3 be the values of the three spins. Assuming the dial is "fair," the probability that of seeing a sum of n is is the number of solutions to $x_1 + x_2 + x_3 = n$ (subject to $x_i \geq 1$), divided by the total number of results.

Using a "pirates and gold bars" approach, there are $\binom{n-1}{n-3}$ solutions to $x + y + z = n$ subject to $x \geq 1, y \geq 1, z \geq 1$. So the answer is

$$\frac{\binom{n-1}{n-3}}{n^3}$$

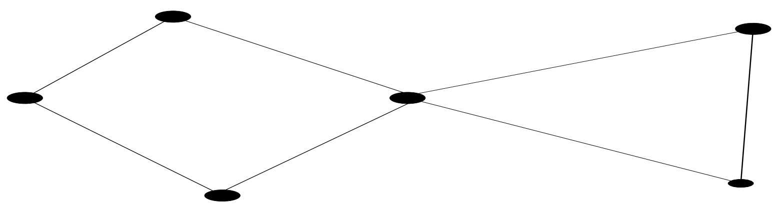
13. [10 points]
Let F be the field of integers modulo 7. Find the degree two, univariate polynomial over F passing through the points $(1, 6)$, $(2, 4)$ and $(3, 4)$.

The polynomial we're looking for has the form $y = ax^2 + bx + c$. We have three unknowns a, b, c, and the three points given. Substituting for (x, y):

- $6 = a + b + c \mod 7$
- $4 = 4a + 2b + c \mod 7$
- $4 = 2a + 3b + c \mod 7$

Solving this system equations (with all arithmetic done modulo 7) gives $a = 1; b = 2; c = 3$ or $y = x^2 + 2x + 3$.

14. [10 points]
Find a simple, Eulerian graph with an even number of vertices and an odd number of edges. Recall that a simple graph has no self loops or multiple edges. An Eulerian graph is a graph with an Euler tour.



15. [10 points]
Prove that every graph with n vertices and $n - k$ edges has at least k components.

We will make use of the fact that in a connected graph, $E \geq V + 1$, where E is the number of edges and V is the number of vertices.

Assume by way of contradiction that the graph has $j < k$ components. Say component i has V_i vertices, in which case (by the above) it has at least $V_i - 1$ edges. So the total number of edges in the graph is $\sum_{i=1}^{j} V_i - 1 = n - j$. But it is given that there are $n - k$ edges in the graph, so we have arrived at a contradiction.