

Bits, Bytes, and Integers – Part 2

15-213: Introduction to Computer Systems
3rd Lecture, Jan. 23, 2018

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First Assignment: Data Lab

- **Due: Thursday, Feb 1, 11:59:00 pm**
- **Last Possible Time to Turn in: Sun, Feb 4, 11:59PM**
- **Read the instructions carefully**
- **Please start ASAP**
- **Seek help (office hours start today!)**
- **Based on Lecture 2, 3 , and 4**
- **After today's lecture you know everything for the integer problems, float problems covered on Thursday**

Summary From Last Lecture

- **Representing information as bits**
- **Bit-level manipulations**
- **Integers**
 - **Representation: unsigned and signed**
 - **Conversion, casting**
 - **Expanding, truncating**
 - **Addition, negation, multiplication, shifting**
- **Representations in memory, pointers, strings**
- **Summary**

Bit-Level Operations in C

■ Operations $\&$, $|$, \sim , \wedge Available in C

- Apply to any “integral” data type
 - long, int, short, char, unsigned
- View arguments as bit vectors
- Arguments applied bit-wise

■ Examples (Char data type)

- $\sim 0x41 \rightarrow 0xBE$
 - $\sim 0100\ 0001_2 \rightarrow 1011\ 1110_2$
- $\sim 0x00 \rightarrow 0xFF$
 - $\sim 0000\ 0000_2 \rightarrow 1111\ 1111_2$
- $0x69 \& 0x55 \rightarrow 0x41$
 - $0110\ 1001_2 \& 0101\ 0101_2 \rightarrow 0100\ 0001_2$
- $0x69 | 0x55 \rightarrow 0x7D$
 - $0110\ 1001_2 | 0101\ 0101_2 \rightarrow 0111\ 1101_2$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Logic Operations in C

■ Logic Operations: `&&`, `||`, `!`

- View 0 as “False”
- Anything nonzero as “True”
- Always return 0 or 1
- Early termination

■ Examples (char data type)

- `!0x41` → `0x00`
- `!0x00` → `0x01`
- `!!0x41` → `0x01`

- `0x69 && 0x55` → `0x01`
- `0x69 || 0x55` → `0x01`
- `p && *p` (avoids null pointer access)

Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit



Two's Complement Examples (w = 5)

		-16	8	4	2	1	
10	=	0	1	0	1	0	

$$8+2 = 10$$

		-16	8	4	2	1	
-10	=	1	0	1	1	0	

$$-16+4+2 = -10$$

Unsigned & Signed Numeric Values

X	$B2U(X)$	$B2T(X)$
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	-8
1001	9	-7
1010	10	-6
1011	11	-5
1100	12	-4
1101	13	-3
1110	14	-2
1111	15	-1

■ Equivalence

- Same encodings for nonnegative values

■ Uniqueness

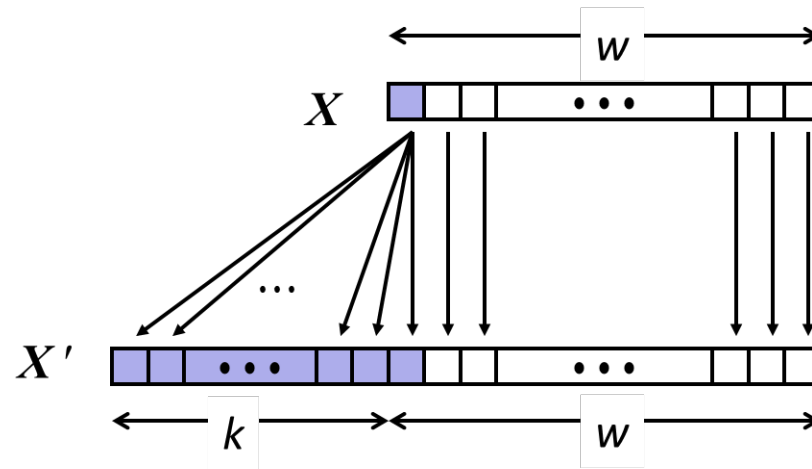
- Every bit pattern represents unique integer value
- Each representable integer has unique bit encoding

■ Expression containing signed and unsigned int:

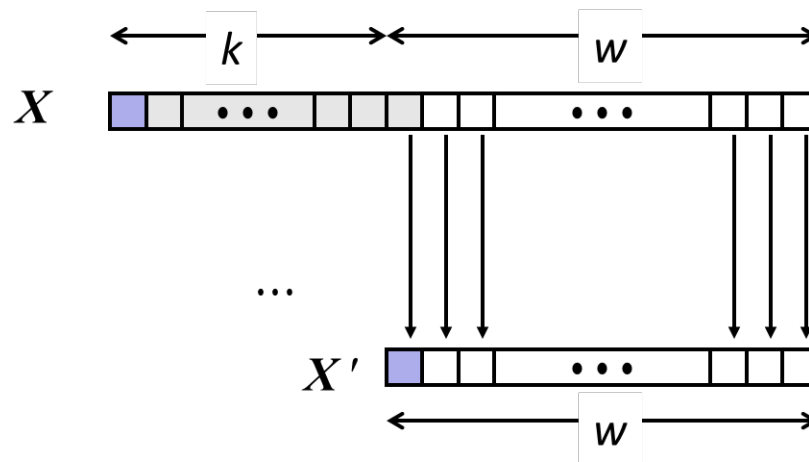
`int` is cast to unsigned!!

Sign Extension and Truncation

■ Sign Extension



■ Truncation



Today: Bits, Bytes, and Integers

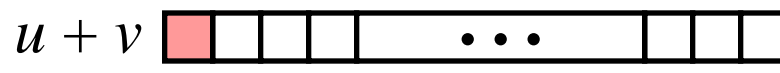
- Representing information as bits
- Bit-level manipulations
- **Integers**
 - Representation: unsigned and signed
 - Conversion, casting
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- Representations in memory, pointers, strings
- Summary

Unsigned Addition

Operands: w bits



True Sum: $w+1$ bits



Discard Carry: w bits



■ Standard Addition Function

- Ignores carry output

■ Implements Modular Arithmetic

$$s = \text{UAdd}_w(u, v) = u + v \bmod 2^w$$

unsigned char	1110 1001	E9	223
	+ 1101 0101	+ D5	+ 213

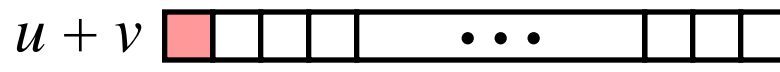
	Hex	Decimal	Binary
0	0	0000	
1	1	0001	
2	2	0010	
3	3	0011	
4	4	0100	
5	5	0101	
6	6	0110	
7	7	0111	
8	8	1000	
9	9	1001	
A	10	1010	
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$$s = \text{UAdd}_w(u, v) = u + v \bmod 2^w$$

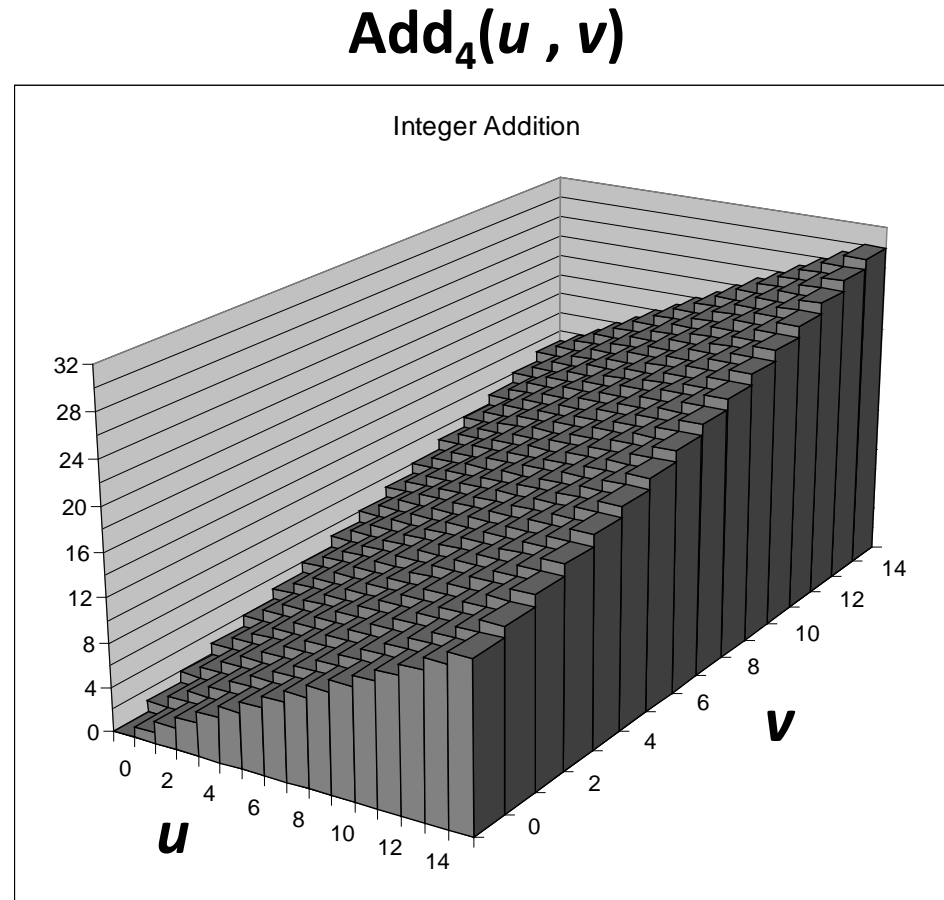
unsigned char	1110 1001	E9	223
	+ 1101 0101	+ D5	+ 213
	1 1011 1110	1BE	446
	1011 1110	BE	190

	Hex	Decimal	Binary
0	0	0000	
1	1	0001	
2	2	0010	
3	3	0011	
4	4	0100	
5	5	0101	
6	6	0110	
7	7	0111	
8	8	1000	
9	9	1001	
A	10	1010	
B	11	1011	
C	12	1100	
D	13	1101	
E	14	1110	
F	15	1111	

Visualizing (**Mathematical**) Integer Addition

■ Integer Addition

- 4-bit integers u, v
- Compute true sum $\text{Add}_4(u, v)$
- Values increase linearly with u and v
- Forms planar surface



Visualizing Unsigned Addition

■ Wraps Around

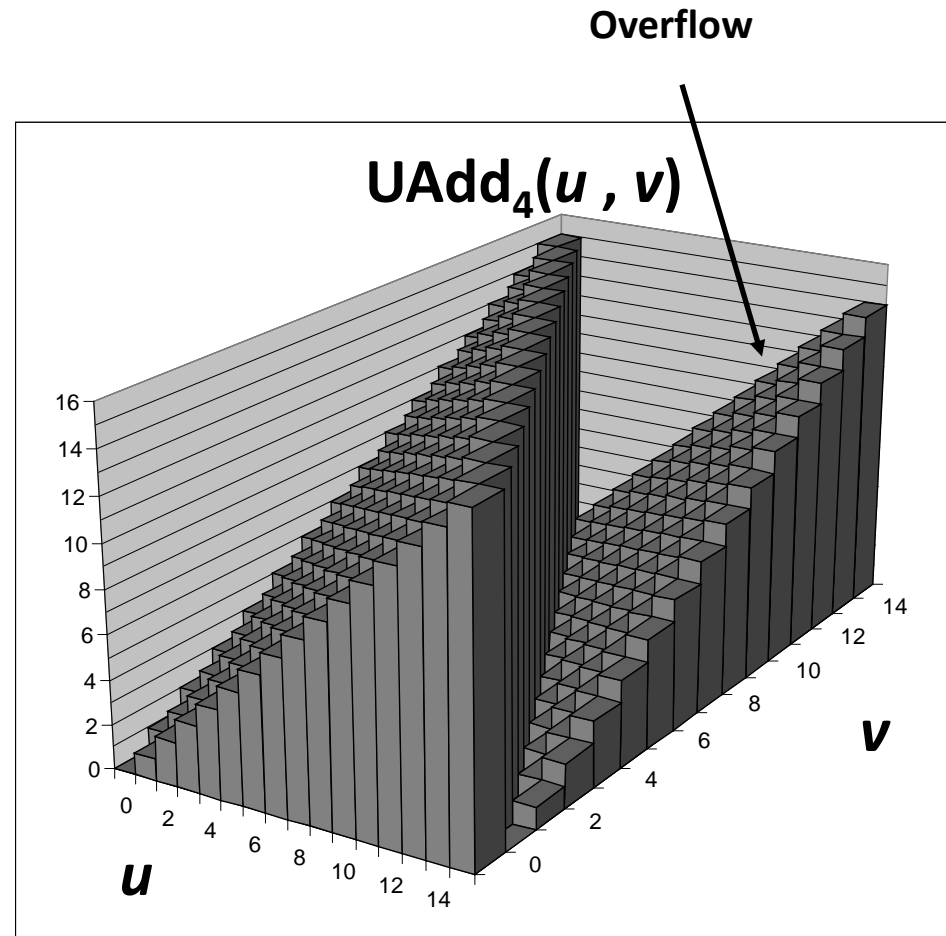
- If true sum $\geq 2^w$
- At most once

True Sum

2^{w+1}
 2^w
 0

Overflow

Modular Sum



Two's Complement Addition

Operands: w bits

u 

+ v 

True Sum: $w+1$ bits

$u + v$ 

Discard Carry: w bits

$\text{TAdd}_w(u, v)$ 

■ TAdd and UAdd have Identical Bit-Level Behavior

- Signed vs. unsigned addition in C:

```
int s, t, u, v;
```

```
s = (int) ((unsigned) u + (unsigned) v);
```

```
t = u + v
```

- Will give `s == t`

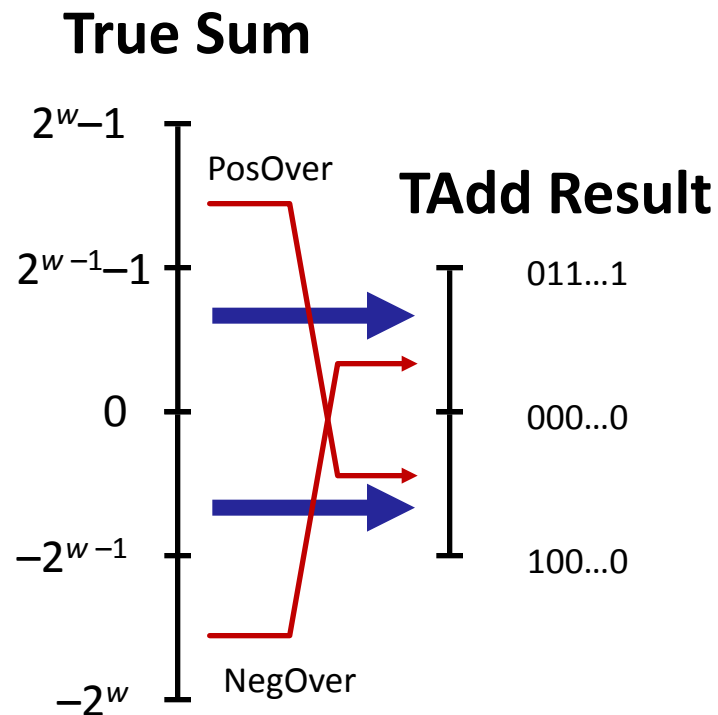
1110 1001	E9	-23
+ 1101 0101	+ D5	+ -43
1 1011 1110	1BE	446
1011 1110	BE	-66

TAdd Overflow

■ Functionality

- True sum requires $w+1$ bits
- Drop off MSB
- Treat remaining bits as 2's comp. integer

0 111...1
 0 100...0
 0 000...0
 1 011...1
 1 000...0



Visualizing 2's Complement Addition

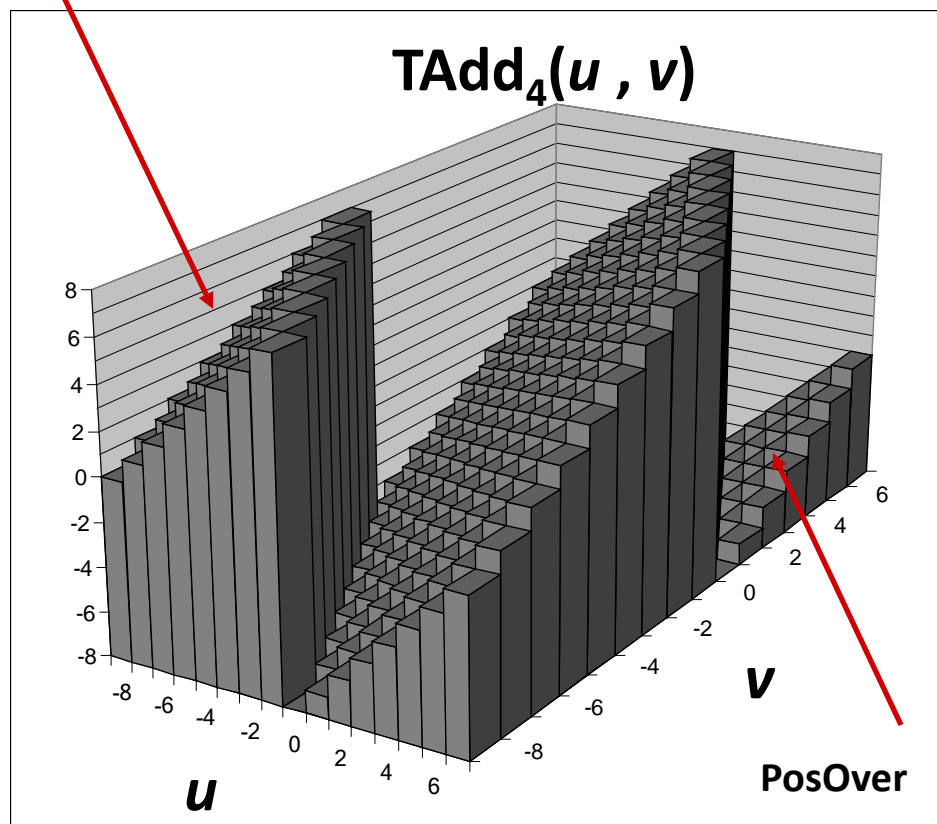
■ Values

- 4-bit two's comp.
- Range from -8 to +7

■ Wraps Around

- If $\text{sum} \geq 2^{w-1}$
 - Becomes negative
 - At most once
- If $\text{sum} < -2^{w-1}$
 - Becomes positive
 - At most once

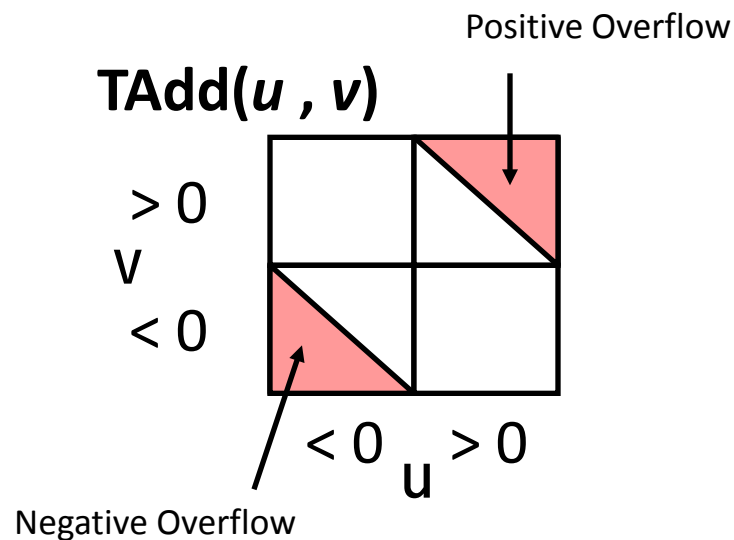
NegOver



Characterizing TAdd

■ Functionality

- True sum requires $w+1$ bits
- Drop off MSB
- Treat remaining bits as 2's comp. integer



$$\mathcal{M}_w(u, v) = \begin{cases} u + v + 2^w & u + v < \mathcal{M}_w \quad (\text{NegOver}) \\ u + v & \mathcal{M}_w \leq u + v \leq \mathcal{M}_w \\ u + v - 2^w & \mathcal{M}_w < u + v \quad (\text{PosOver}) \end{cases}$$

Negation: Complement & Increment

■ Negate through complement and increase

$$\sim x + 1 == -x$$

■ Example

- Observation: $\sim x + x == 1111\dots111 == -1$

$$\begin{array}{r}
 x \quad \boxed{10011101} \\
 + \quad \sim x \quad \boxed{01100010} \\
 \hline
 -1 \quad \boxed{11111111}
 \end{array}$$

x = 15213

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
~x	-15214	C4 92	11000100 10010010
~x+1	-15213	C4 93	11000100 10010011
y	-15213	C4 93	11000100 10010011

Complement & Increment Examples

$x = 0$

	Decimal	Hex	Binary
0	0	00 00	00000000 00000000
~ 0	-1	FF FF	11111111 11111111
$\sim 0 + 1$	0	00 00	00000000 00000000

$x = \text{TMin}$

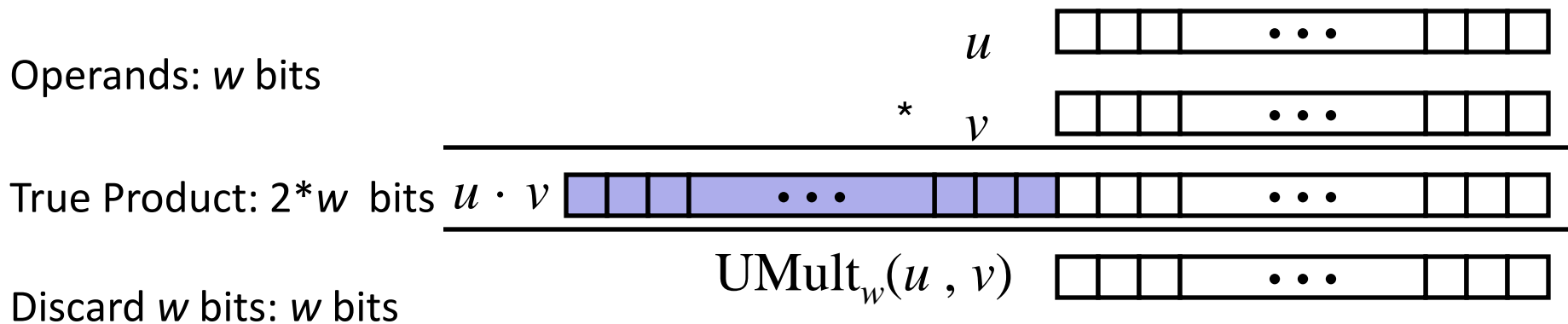
	Decimal	Hex	Binary
x	-32768	80 00	10000000 00000000
$\sim x$	32767	7F FF	01111111 11111111
$\sim x + 1$	-32768	80 00	10000000 00000000

Canonical counter example

Multiplication

- **Goal: Computing Product of w -bit numbers x, y**
 - Either signed or unsigned
- **But, exact results can be bigger than w bits**
 - Unsigned: up to $2w$ bits
 - Result range: $0 \leq x * y \leq (2^w - 1)^2 = 2^{2w} - 2^{w+1} + 1$
 - Two's complement min (negative): Up to $2w-1$ bits
 - Result range: $x * y \geq (-2^{w-1}) * (2^{w-1} - 1) = -2^{2w-2} + 2^{w-1}$
 - Two's complement max (positive): Up to $2w$ bits, but only for $(TMin_w)^2$
 - Result range: $x * y \leq (-2^{w-1})^2 = 2^{2w-2}$
- **So, maintaining exact results...**
 - would need to keep expanding word size with each product computed
 - is done in software, if needed
 - e.g., by “arbitrary precision” arithmetic packages

Unsigned Multiplication in C



■ Standard Multiplication Function

- Ignores high order w bits

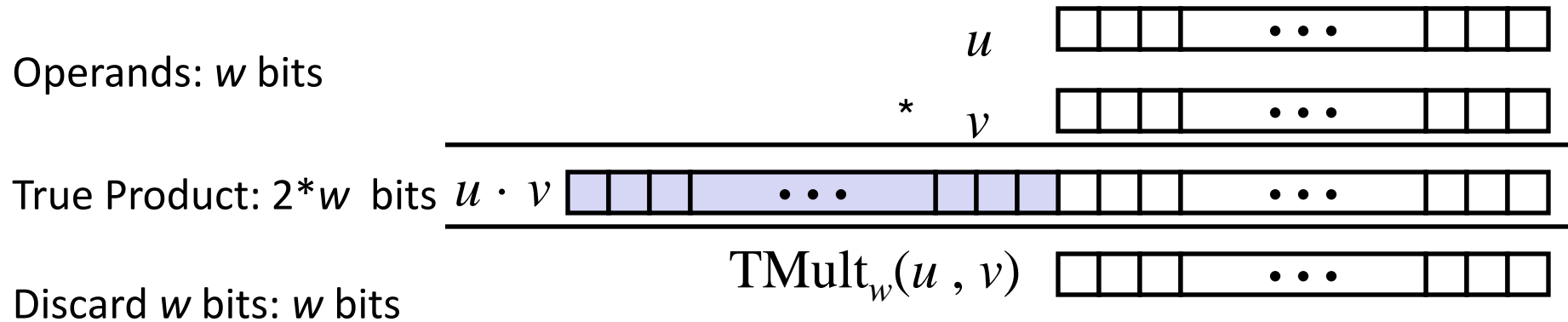
■ Implements Modular Arithmetic

$$\text{UMult}_w(u, v) = u \cdot v \bmod 2^w$$

$$\begin{array}{r}
 1110 \ 1001 \\
 * \ 1101 \ 0101 \\
 \hline
 1100 \ 0001 \ 1101 \ 0010 \\
 \hline
 1101 \ 1101
 \end{array}$$

$$\begin{array}{r}
 \text{E9} \quad 223 \\
 * \ \text{D5} \quad 213 \\
 \hline
 \text{C1DD} \quad 47499 \\
 \hline
 \text{DD} \quad 221
 \end{array}$$

Signed Multiplication in C



■ Standard Multiplication Function

- Ignores high order w bits
- Some of which are different for signed vs. unsigned multiplication
- Lower bits are the same

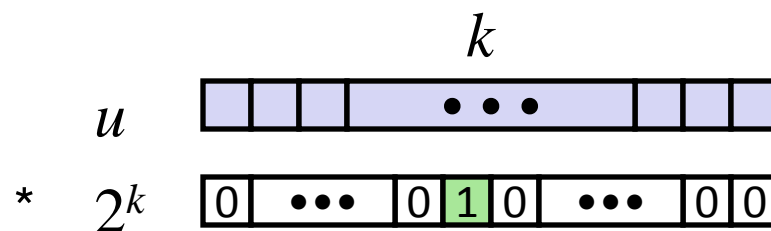
	1110 1001	E9	-23
*	1101 0101	* D5	* -43
	1100 0001 1101 0010	C1DD	16896
	1101 1101	DD	-35

Power-of-2 Multiply with Shift

■ Operation

- $u \ll k$ gives $u * 2^k$
- Both signed and unsigned

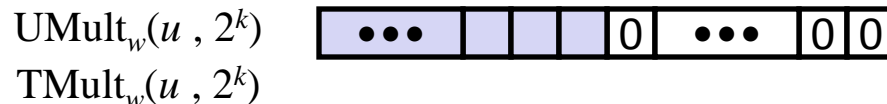
Operands: w bits



True Product: $w+k$ bits



Discard k bits: w bits



■ Examples

- $u \ll 3 \quad == \quad u * 8$
- $(u \ll 5) - (u \ll 3) == \quad u * 24$
- Most machines shift and add faster than multiply
 - Compiler generates this code automatically

Break Time!

taradiddle: "pretentious nonsense"

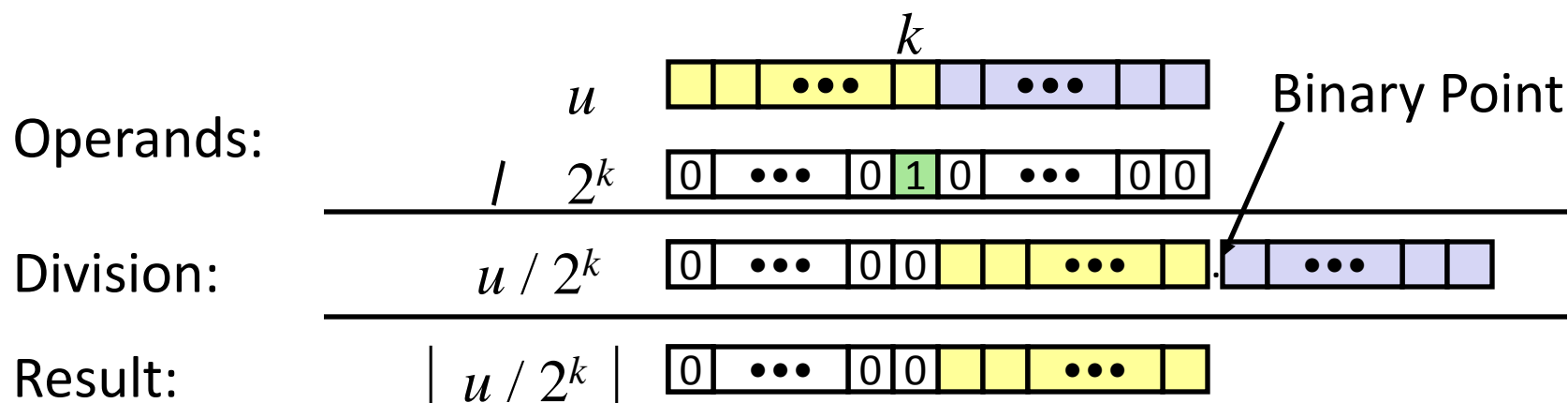
Check out: quiz: day 3: operators

<https://canvas.cmu.edu/courses/3822/quizzes/9020>

Unsigned Power-of-2 Divide with Shift

■ Quotient of Unsigned by Power of 2

- $u \gg k$ gives $\lfloor u / 2^k \rfloor$
- Uses logical shift

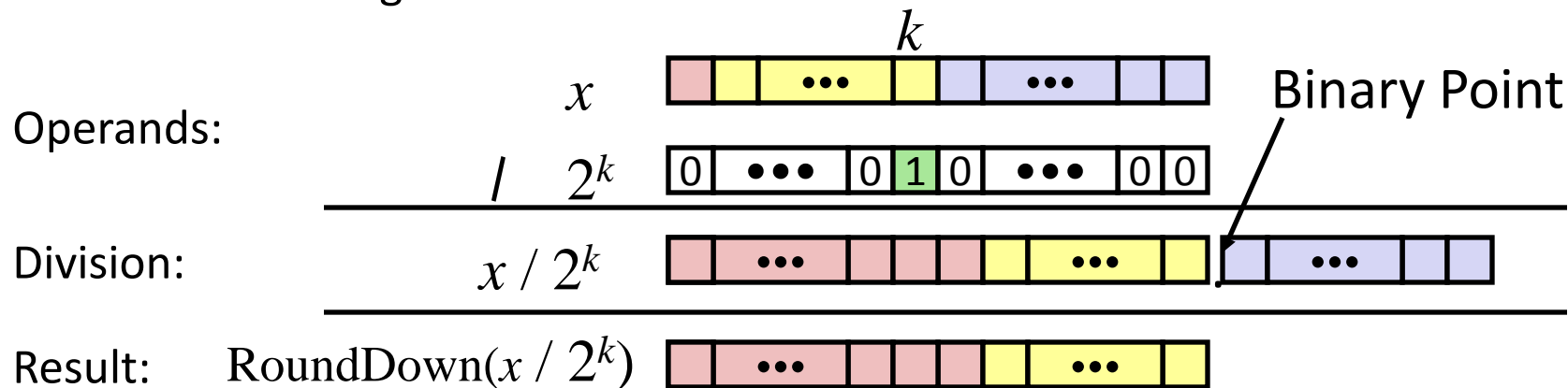


	Division	Computed	Hex	Binary
x	15213	15213	3B 6D	00111011 01101101
x >> 1	7606.5	7606	1D B6	00011101 10110110
x >> 4	950.8125	950	03 B6	00000011 10110110
x >> 8	59.4257813	59	00 3B	00000000 00111011

Signed Power-of-2 Divide with Shift

■ Quotient of Signed by Power of 2

- $x \gg k$ gives $\lfloor x / 2^k \rfloor$
- Uses arithmetic shift
- Rounds wrong direction when $u < 0$



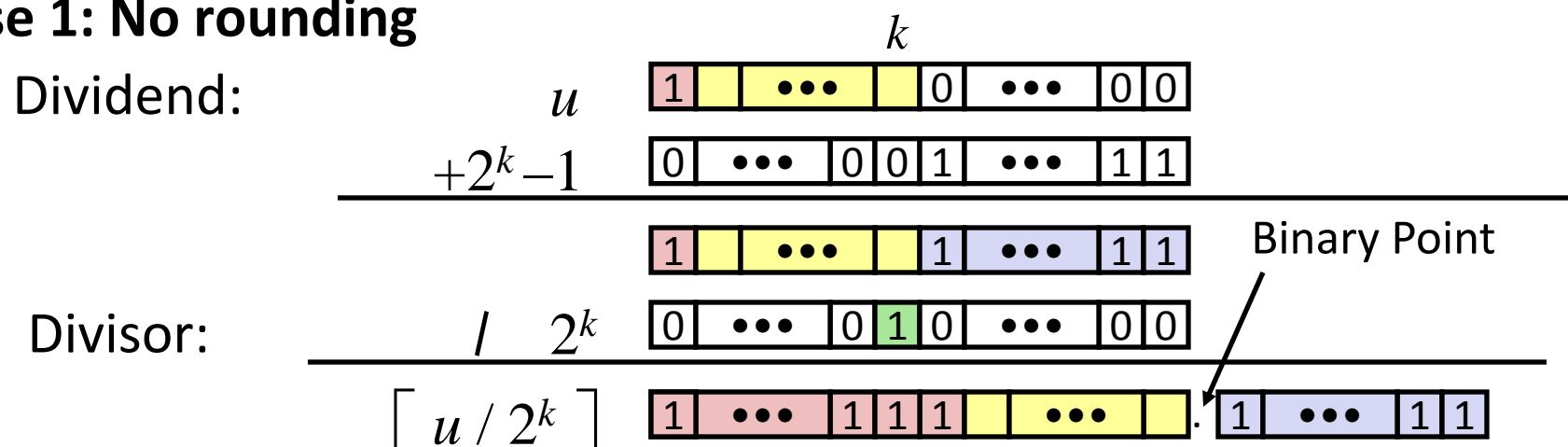
	Division	Computed	Hex	Binary
y	-15213	-15213	C4 93	11000100 10010011
$y \gg 1$	-7606.5	-7607	E2 49	11100010 01001001
$y \gg 4$	-950.8125	-951	FC 49	11111100 01001001
$y \gg 8$	-59.4257813	-60	FF C4	11111111 11000100

Correct Power-of-2 Divide

■ Quotient of Negative Number by Power of 2

- Want $\lceil \mathbf{x} / 2^k \rceil$ (Round Toward 0)
- Compute as $\lfloor (\mathbf{x} + 2^k - 1) / 2^k \rfloor$
 - In C: $(\mathbf{x} + (1 \ll k) - 1) \gg k$
 - Biases dividend toward 0

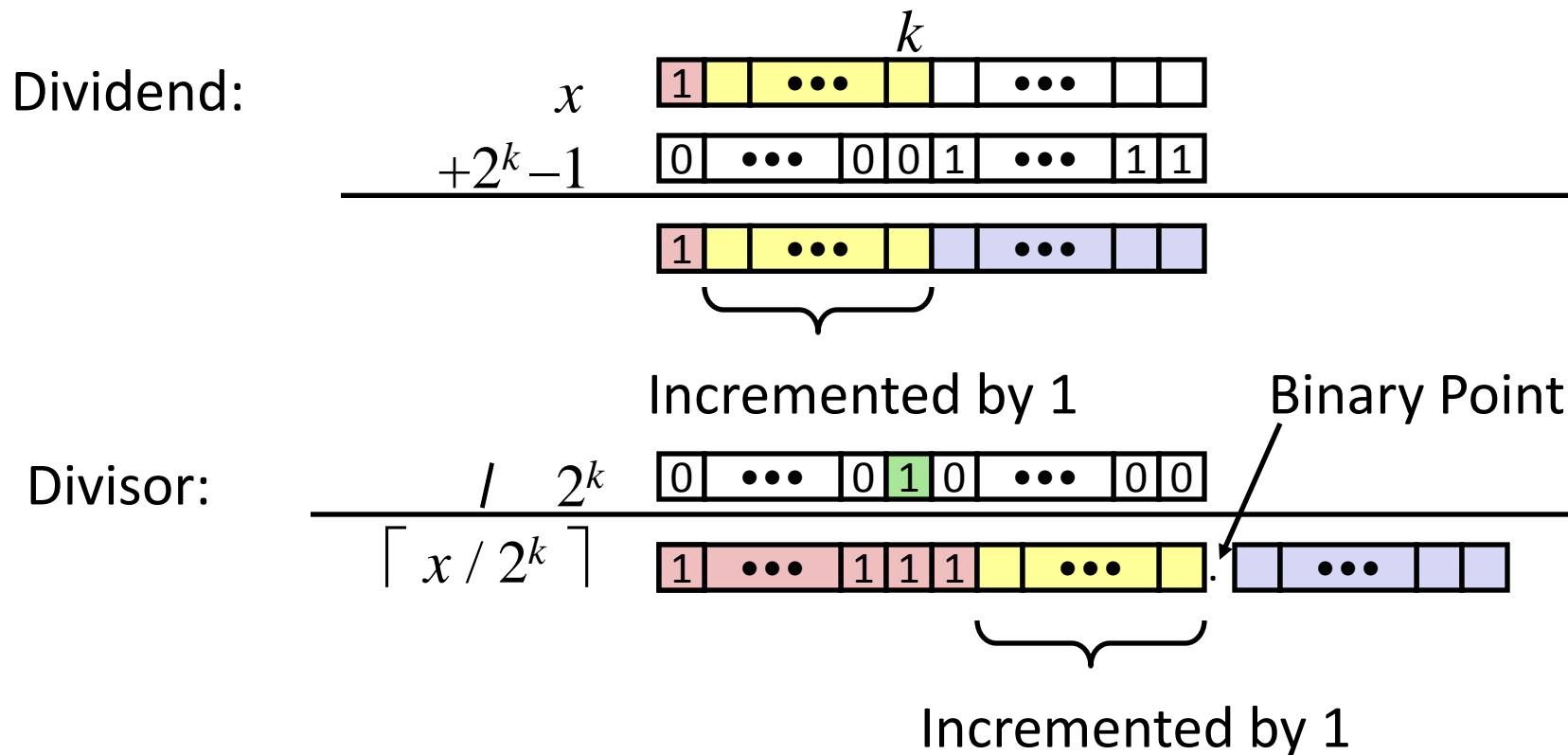
Case 1: No rounding



Biassing has no effect

Correct Power-of-2 Divide (Cont.)

Case 2: Rounding



Biasing adds 1 to final result

Today: Bits, Bytes, and Integers

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Arithmetic: Basic Rules

■ Addition:

- Unsigned/signed: Normal addition followed by truncate, same operation on bit level
- Unsigned: addition mod 2^w
 - Mathematical addition + possible subtraction of 2^w
- Signed: modified addition mod 2^w (result in proper range)
 - Mathematical addition + possible addition or subtraction of 2^w

■ Multiplication:

- Unsigned/signed: Normal multiplication followed by truncate, same operation on bit level
- Unsigned: multiplication mod 2^w
- Signed: modified multiplication mod 2^w (result in proper range)

Why Should I Use Unsigned?

■ *Don't* use without understanding implications

- Easy to make mistakes

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
    a[i] += a[i+1];
```

- Can be very subtle

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
    . . .
```

Counting Down with Unsigned

■ Proper way to use unsigned as loop index

```
unsigned i;  
for (i = cnt-2; i < cnt; i--)  
    a[i] += a[i+1];
```

■ See Robert Seacord, *Secure Coding in C and C++*

- C Standard guarantees that unsigned addition will behave like modular arithmetic
 - $0 - 1 \rightarrow UMax$

■ Even better

```
size_t i;  
for (i = cnt-2; i < cnt; i--)  
    a[i] += a[i+1];
```

- Data type `size_t` defined as unsigned value with length = word size
- Code will work even if `cnt = UMax`
- What if `cnt` is signed and `< 0`?

Why Should I Use Unsigned? (cont.)

- **Do Use When Performing Modular Arithmetic**
 - Multiprecision arithmetic
- **Do Use When Using Bits to Represent Sets**
 - Logical right shift, no sign extension
- **Do Use In System Programming**
 - Bit masks, device commands,...

Integer Arithmetic Example

unsigned char

1111 0011	F3	243
+ 0101 0010	+ 52	+ 82
1 0100 0101	145	325
0101 0101	45	69

unsigned char

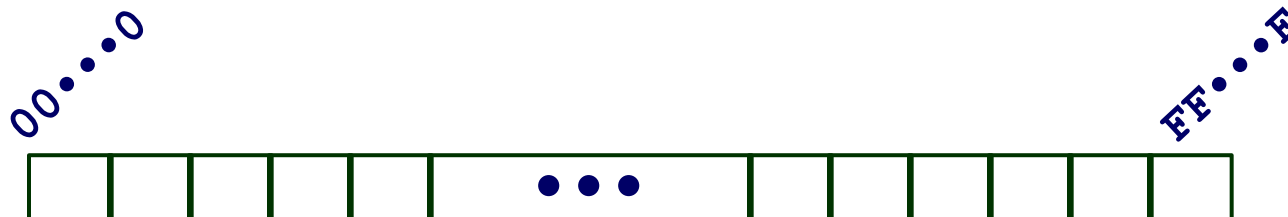
0001 1001	19	25
* 0000 0010	* 02	* 2
0 0011 0010	032	50
0011 0010	32	50

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
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Byte-Oriented Memory Organization



■ Programs refer to data by address

- Conceptually, envision it as a very large array of bytes
 - In reality, it's not, but can think of it that way
- An address is like an index into that array
 - and, a pointer variable stores an address

■ Note: system provides private address spaces to each “process”

- Think of a process as a program being executed
- So, a program can clobber its own data, but not that of others

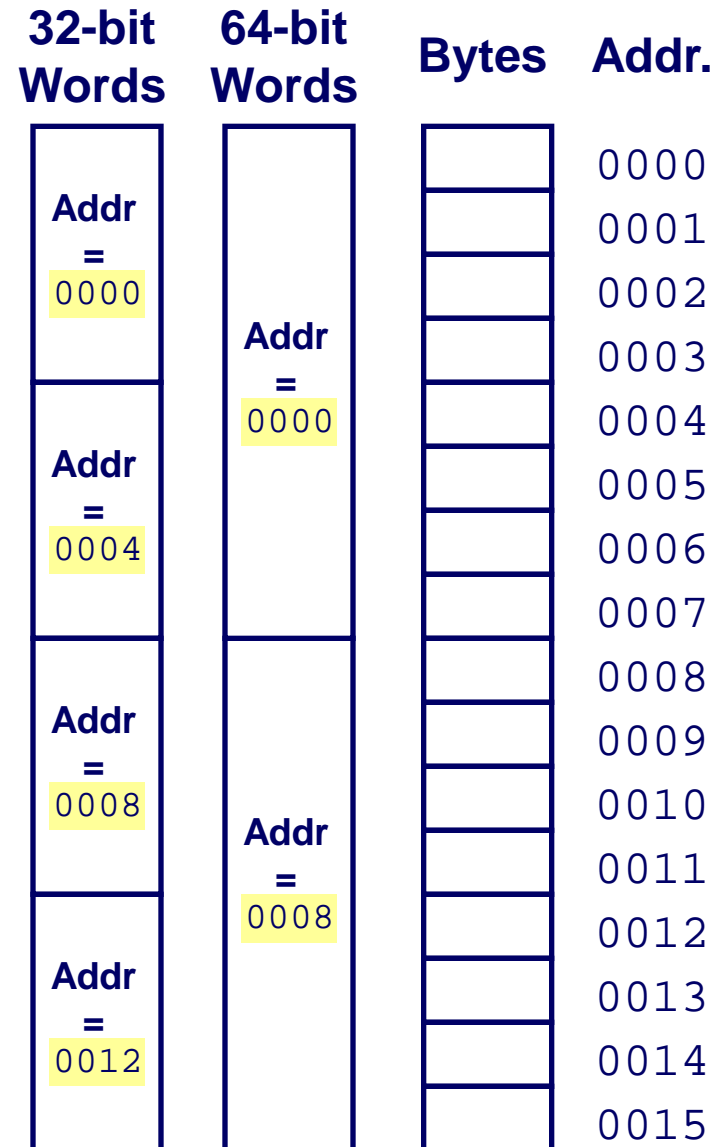
Machine Words

- **Any given computer has a “Word Size”**
 - Nominal size of integer-valued data
 - and of addresses
 - Until recently, most machines used 32 bits (4 bytes) as word size
 - Limits addresses to 4GB (2^{32} bytes)
 - Increasingly, machines have 64-bit word size
 - Potentially, could have 18 EB (exabytes) of addressable memory
 - That's 18.4×10^{18}
 - Machines still support multiple data formats
 - Fractions or multiples of word size
 - Always integral number of bytes

Word-Oriented Memory Organization

■ Addresses Specify Byte Locations

- Address of first byte in word
- Addresses of successive words differ by 4 (32-bit) or 8 (64-bit)



Example Data Representations

C Data Type	Typical 32-bit	Typical 64-bit	x86-64
<code>char</code>	1	1	1
<code>short</code>	2	2	2
<code>int</code>	4	4	4
<code>long</code>	4	8	8
<code>float</code>	4	4	4
<code>double</code>	8	8	8
<code>pointer</code>	4	8	8

Byte Ordering

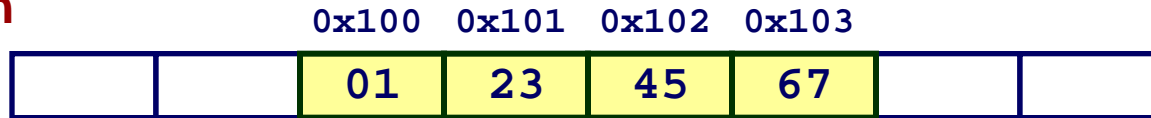
- So, how are the bytes within a multi-byte word ordered in memory?
- Conventions
 - Big Endian: Sun, PPC Mac, *Internet*
 - Least significant byte has highest address
 - Little Endian: *x86*, ARM processors running Android, iOS, and Windows
 - Least significant byte has lowest address

Byte Ordering Example

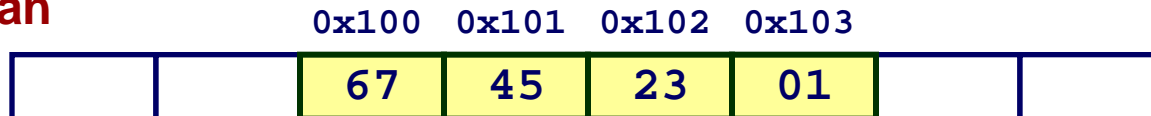
■ Example

- Variable x has 4-byte value of 0x01234567
- Address given by &x is 0x100

Big Endian



Little Endian



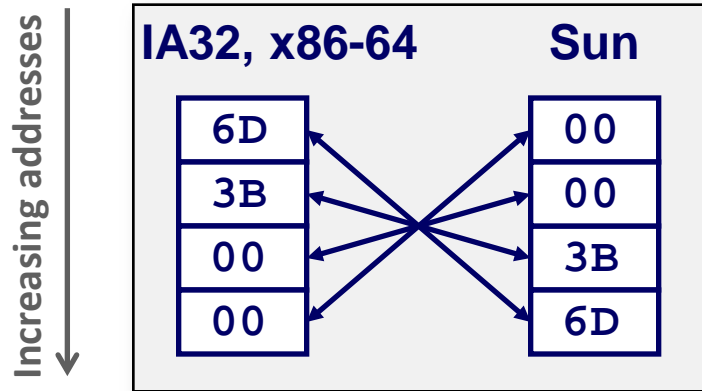
Representing Integers

Decimal: 15213

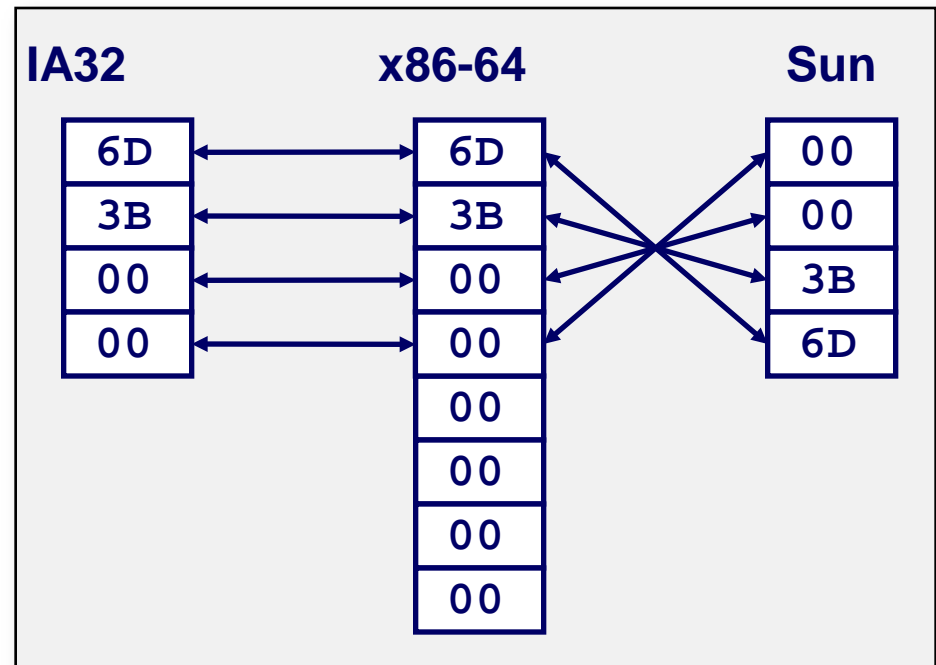
Binary: 0011 1011 0110 1101

Hex: 3 B 6 D

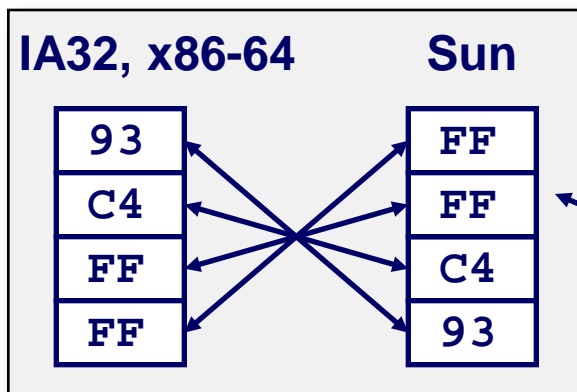
`int A = 15213;`



`long int C = 15213;`



`int B = -15213;`



Two's complement representation

Examining Data Representations

■ Code to Print Byte Representation of Data

- Casting pointer to unsigned char * allows treatment as a byte array

```
typedef unsigned char *pointer;

void show_bytes(pointer start, size_t len){
    size_t i;
    for (i = 0; i < len; i++)
        printf("%p\t0x%.2x\n",start+i, start[i]);
    printf("\n");
}
```

Printf directives:

%p: Print pointer

%x: Print Hexadecimal

show_bytes Execution Example

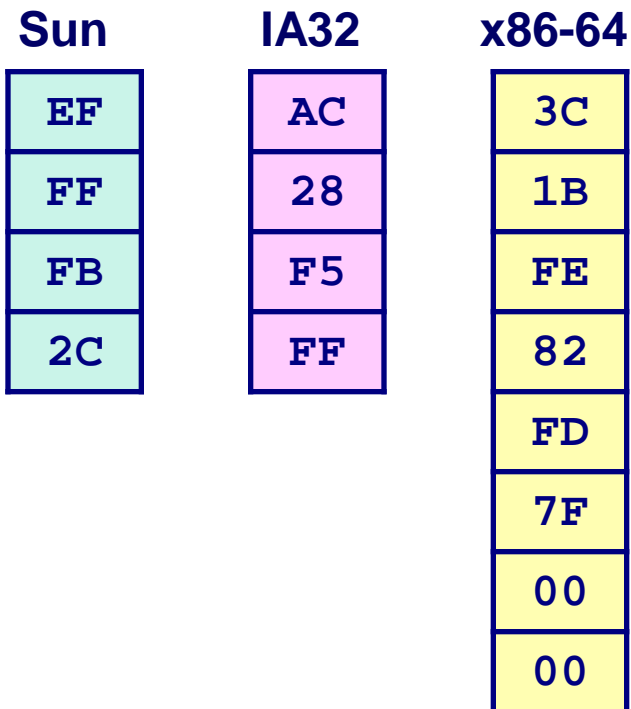
```
int a = 15213;
printf("int a = 15213;\n");
show_bytes((pointer) &a, sizeof(int));
```

Result (Linux x86-64):

```
int a = 15213;
0x7ffffb7f71dbc    6d
0x7ffffb7f71dbd    3b
0x7ffffb7f71dbe    00
0x7ffffb7f71dbf    00
```

Representing Pointers

```
int B = -15213;  
int *P = &B;
```



Different compilers & machines assign different locations to objects

Even get different results each time run program

Representing Strings

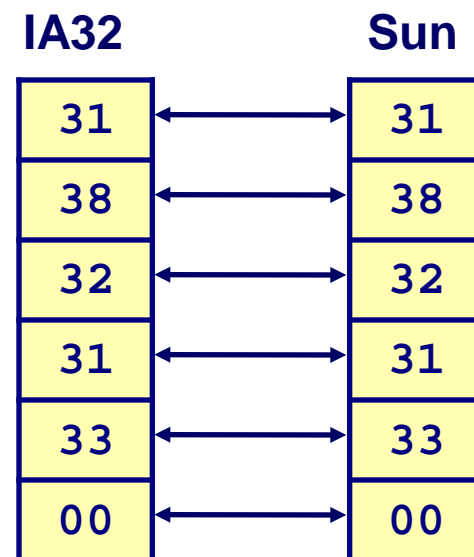
```
char S[6] = "18213";
```

■ Strings in C

- Represented by array of characters
- Each character encoded in ASCII format
 - Standard 7-bit encoding of character set
 - Character "0" has code 0x30
 - Digit i has code $0x30+i$
- String should be null-terminated
 - Final character = 0

■ Compatibility

- Byte ordering not an issue



Reading Byte-Reversed Listings

■ Disassembly

- Text representation of binary machine code
- Generated by program that reads the machine code

■ Example Fragment

Address	Instruction Code	Assembly Rendition
8048365:	5b	pop %ebx
8048366:	81 c3 ab 12 00 00	add \$0x12ab,%ebx
804836c:	83 bb 28 00 00 00 00	cmpl \$0x0,0x28(%ebx)

■ Deciphering Numbers

- Value: 0x12ab
- Pad to 32 bits: 0x000012ab
- Split into bytes: 00 00 12 ab
- Reverse: ab 12 00 00

Integer C Puzzles

Initialization

```
int x = foo();
int y = bar();
unsigned ux = x;
unsigned uy = y;
```

`x < 0` \Rightarrow `((x*2) < 0)`

`ux >= 0`

`x & 7 == 7` \Rightarrow `(x<<30) < 0`

`ux > -1`

`x > y` \Rightarrow `-x < -y`

`x * x >= 0`

`x > 0 && y > 0` \Rightarrow `x + y > 0`

`x >= 0` \Rightarrow `-x <= 0`

`x <= 0` \Rightarrow `-x >= 0`

`(x|-x)>>31 == -1`

`ux >> 3 == ux/8`

`x >> 3 == x/8`

`x & (x-1) != 0`

Integer C Puzzles

Initialization

```
int x = foo();
int y = bar();
unsigned ux = x;
unsigned uy = y;
```

<code>x < 0</code>	\Rightarrow	<code>((x*2) < 0)</code>	✗
<code>ux >= 0</code>			✓
<code>x & 7 == 7</code>	\Rightarrow	<code>(x<<30) < 0</code>	✓
<code>ux > -1</code>			✗
<code>x > y</code>	\Rightarrow	<code>-x < -y</code>	✗
<code>x * x >= 0</code>			✗
<code>x > 0 && y > 0</code>	\Rightarrow	<code>x + y > 0</code>	✗
<code>x >= 0</code>	\Rightarrow	<code>-x <= 0</code>	✓
<code>x <= 0</code>	\Rightarrow	<code>-x >= 0</code>	✗
<code>(x -x)>>31 == -1</code>			✗
<code>ux >> 3 == ux/8</code>			✓
<code>x >> 3 == x/8</code>			✗
<code>x & (x-1) != 0</code>			✗

Summary

- **Representing information as bits**
- **Bit-level manipulations**
- **Integers**
 - **Representation: unsigned and signed**
 - **Conversion, casting**
 - **Expanding, truncating**
 - **Addition, negation, multiplication, shifting**
- **Representations in memory, pointers, strings**
- **Summary**