15-213

"The course that gives CMU its Zip!"

Floating Point Arithmetic
February 15, 2001

Topics

- **IEEE Floating Point Standard**
- **Rounding**
- **Floating Point Operations**
- **Mathematical properties**
- **IA32 floating point**

Floating Point Puzzles

- **For each of the following C expressions, either:**
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither d nor f is NaN

- `x == (int)(float) x`
- `x == (int)(double) x`
- `f == (float)(double) f`
- `d == (float) d`
- `f == -(-f);`
- `2/3 == 2/3.0`
- `d < 0.0` ⇒ `((d*2) < 0.0)`
- `d > f` ⇒ `-f < -d`
- `d * d >= 0.0`
- `(d+f)-d == f`

IEEE Floating Point

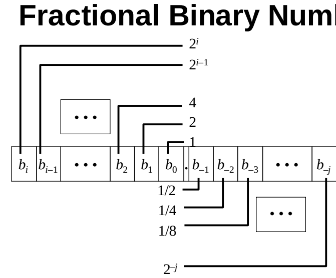
IEEE Standard 754

- **Established in 1985 as uniform standard for floating point arithmetic**
  - Before that, many idiosyncratic formats
- **Supported by all major CPUs**

Driven by Numerical Concerns

- **Nice standards for rounding, overflow, underflow**
- **Hard to make go fast**
  - Numerical analysts predominated over hardware types in defining standard

Fractional Binary Numbers



Representation

- **Bits to right of "binary point" represent fractional powers of 2**
- **Represents rational number:** $\sum_{k=-j}^{i} b_k \cdot 2^k$

Fractional Binary Number Examples

Value — **Representation**

Value	Representation
5-3/4	`101.11`$_2$
2-7/8	`10.111`$_2$
63/64	`0.111111`$_2$

Observation

- **Divide by 2 by shifting right**
- **Numbers of form `0.111111…`$_2$ just below 1.0**
  - Use notation 1.0 – ε

Limitation

- **Can only exactly represent numbers of the form $x/2^k$**
- **Other numbers have repeating bit representations**

Value	Representation
1/3	`0.0101010101[01]…`$_2$
1/5	`0.001100110011[0011]…`$_2$
1/10	`0.0001100110011[0011]…`$_2$

Floating Point Representation

Numerical Form

- $-1^s\ M\ 2^E$
  - Sign bit *s* determines whether number is negative or positive
  - Significand *M* normally a fractional value in range [1.0,2.0).
  - Exponent *E* weights value by power of two

Encoding

s	exp	frac

- **MSB is sign bit**
- **`exp` field encodes *E***
- **`frac` field encodes *M***

Sizes

- **Single precision: 8 `exp` bits, 23 `frac` bits**
  - 32 bits total
- **Double precision: 11 `exp` bits, 52 `frac` bits**
  - 64 bits total

"Normalized" Numeric Values

Condition

- **`exp` ≠ `000…0` and `exp` ≠ `111…1`**

Exponent coded as *biased* value

E* = *Exp* – *Bias

- *Exp* : unsigned value denoted by **`exp`**
- *Bias* : Bias value
  - » Single precision: 127 (*Exp*: 1…254, *E*: -126…127)
  - » Double precision: 1023 (*Exp*: 1…2046, *E*: -1022…1023)
  - » in general: *Bias* = 2^{m-1} - 1, where m is the number of exponent bits

Significand coded with implied leading 1

***m* = `1.xxx…x`$_2$**

- **`xxx…x`**: bits of `frac`
- Minimum when **`000…0`** (*M* = 1.0)
- Maximum when **`111…1`** (*M* = 2.0 – ε)
- Get extra leading bit for "free"

Normalized Encoding Example

Value

`Float F = 15213.0;`

- 15213_{10} = 11101101101101_2 = 1.1101101101101_2 X 2^{13}

Significand

M = $1.\underline{1101101101101}_2$

`frac` = **$\underline{1101101101101}0000000000_2$**

Exponent

E = **13**

Bias = **127**

Exp = **140** = **10001100_2**

Floating Point Representation (Class 02):

Hex:	**4**	**6**	**6**	**D**	**B**	**4**	**0**	**0**
Binary:	**0100**	**0110**	**0110**	**1101**	**1011**	**0100**	**0000**	**0000**
140:	**100**	**0110**	**0**					
15213:			***1*110**	**1101**	**1011**	**01**		

Denormalized Values

Condition

- **`exp` = `000…0`**

Value

- **Exponent value $E = -Bias + 1$**
- **Significand value $m =$ `0.xxx…x`$_2$**
  - **`xxx…x`:** bits of `frac`

Cases

- **`exp` = `000…0`, `frac` = `000…0`**
  - Represents value 0
  - Note that have distinct values +0 and –0
- **`exp` = `000…0`, `frac` ≠ `000…0`**
  - Numbers very close to 0.0
  - Lose precision as get smaller
  - "Gradual underflow"

Special Values

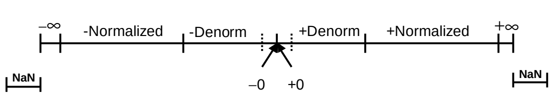
Condition

- **`exp` = `111…1`**

Cases

- **`exp` = `111…1`, `frac` = `000…0`**
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = –1.0/–0.0 = +∞, 1.0/–0.0 = –∞
- **`exp` = `111…1`, `frac` ≠ `000…0`**
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ – ∞

Summary of Floating Point Real Number Encodings

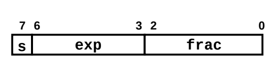


Tiny floating point example

8-bit Floating Point Representation

- **the sign bit is in the most significant bit.**
- **the next four bits are the exponent, with a bias of 7.**
- **the last three bits are the `frac`**

- **Same General Form as IEEE Format**
  - **normalized, denormalized**
  - **representation of 0, NaN, infinity**



Values related to the exponent

Exp	exp	E	2^E	
0	0000	-6	1/64	(denorms)
1	0001	-6	1/64	
2	0010	-5	1/32	
3	0011	-4	1/16	
4	0100	-3	1/8	
5	0101	-2	1/4	
6	0110	-1	1/2	
7	0111	0	1	
8	1000	+1	2	
9	1001	+2	4	
10	1010	+3	8	
11	1011	+4	16	
12	1100	+5	32	
13	1101	+6	64	
14	1110	+7	128	
15	1111	n/a		(inf, NaN)

Dynamic Range

	s exp frac	*E*	Value	
	0 0000 000	-6	0	
	0 0000 001	-6	1/8*1/64 = 1/512	← closest to zero
Denormalized numbers	0 0000 010	-6	2/8*1/64 = 2/512	
	…			
	0 0000 110	-6	6/8*1/64 = 6/512	
	0 0000 111	-6	7/8*1/64 = 7/512	← largest denorm
	0 0001 000	-6	8/8*1/64 = 8/512	← smallest norm
	0 0001 001	-6	9/8*1/64 = 9/512	
	…			
	0 0110 110	-1	14/8*1/2 = 14/16	
	0 0110 111	-1	15/8*1/2 = 15/16	← closest to 1 below
Normalized numbers	0 0111 000	0	8/8*1 = 1	
	0 0111 001	0	9/8*1 = 9/8	← closest to 1 above
	0 0111 010	0	10/8*1 = 10/8	
	…			
	0 1110 110	7	14/8*128 = 224	
	0 1110 111	7	15/8*128 = 240	← largest norm
	0 1111 000	n/a	inf	

Interesting Numbers

Description	exp	frac	Numeric Value
Zero	00…00	00…00	0.0
Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
• Single ≈ 1.4×10^{-45}			
• Double ≈ 4.9×10^{-324}			
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
• Single ≈ 1.18×10^{-38}			
• Double ≈ 2.2×10^{-308}			
Smallest Pos. Normalized	00…01	00…00	$1.0 \times 2^{-\{126,1022\}}$
• Just larger than largest denormalized			
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
• Single ≈ 3.4×10^{38}			
• Double ≈ 1.8×10^{308}			

Special Properties of Encoding

FP Zero Same as Integer Zero

- **All bits = 0**

Can (Almost) Use Unsigned Integer Comparison

- **Must first compare sign bits**
- **Must consider -0 = 0**
- **NaNs problematic**
  - Will be greater than any other values
  - What should comparison yield?
- **Otherwise OK**
  - Denorm vs. normalized
  - Normalized vs. infinity

Floating Point Operations

Conceptual View

- First compute exact result
- Make it fit into desired precision
  - Possibly overflow if exponent too large
  - Possibly round to fit into `frac`

Rounding Modes (illustrate with $ rounding)

	$1.40	$1.60	$1.50	$2.50	–$1.50
• Zero	$1.00	$1.00	$1.00	$2.00	–$1.00
• Round down (-∞)	$1.00	$1.00	$1.00	$2.00	–$2.00
• Round up (+∞)	$2.00	$2.00	$2.00	$3.00	–$1.00
• Nearest Even (default)	$1.00	$2.00	$2.00	$2.00	–$2.00

Note:
1. Round down: rounded result is close to but no greater than true result.
2. Round up: rounded result is close to but no less than true result.

A Closer Look at Round-To-Even

Default Rounding Mode

- Hard to get any other kind without dropping into assembly
- All others are statistically biased
  - Sum of set of positive numbers will consistently be over- or under-estimated

Applying to Other Decimal Places

- When exactly halfway between two possible values
  - Round so that least significant digit is even
- E.g., round to nearest hundredth

1.2349999	1.23	(Less than half way)
1.2350001	1.24	(Greater than half way)
1.2350000	1.24	(Half way—round up)
1.2450000	1.24	(Half way—round down)

Rounding Binary Numbers

Binary Fractional Numbers

- "Even" when least significant bit is `0`
- Half way when bits to right of rounding position = $100\ldots_2$

Examples

- Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

Operands

$(-1)^{s1}\ M1\ \ 2^{E1}$

$(-1)^{s2}\ M2\ \ 2^{E2}$

Exact Result

$(-1)^{s}\ M\ \ 2^{E}$

- **Sign** *s*: *s1 ^ s2*
- **Significand** *M*: *M1 * M2*
- **Exponent** *E*: *E1 + E2*

Fixing

- If $M \geq 2$, shift M right, increment E
- If E out of range, overflow
- Round M to fit `frac` precision

Implementation

- Biggest chore is multiplying significands

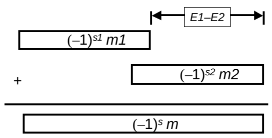
FP Addition

Operands

$(-1)^{s1}\ M1\ \ 2^{E1}$

$(-1)^{s2}\ M2\ \ 2^{E2}$

- **Assume** *E1* **>** *E2*



Exact Result

$(-1)^{s}\ M\ \ 2^{E}$

- **Sign** *s*, **significand** *M***:**
  - Result of signed align & add
- **Exponent** *E***:** *E1*

Fixing

- **If** $M \geq 2$, **shift** *M* **right, increment** *E*
- **if** $M < 1$, **shift** *M* **left** *k* **positions, decrement** *E* **by** *k*
- **Overflow if** *E* **out of range**
- **Round** *M* **to fit `frac` precision**

Mathematical Properties of FP Add

Compare to those of Abelian Group

- **Closed under addition?** **YES**
  - But may generate infinity or NaN
- **Commutative?** **YES**
- **Associative?** **NO**
  - Overflow and inexactness of rounding
- **0 is additive identity?** **YES**
- **Every element has additive inverse** **ALMOST**
  - Except for infinities & NaNs

Montonicity

- $a \geq b \Rightarrow a+c \geq b+c$**?** **ALMOST**
  - Except for infinities & NaNs

Algebraic Properties of FP Mult

Compare to Commutative Ring

- **Closed under multiplication?** **YES**
  - But may generate infinity or NaN
- **Multiplication Commutative?** **YES**
- **Multiplication is Associative?** **NO**
  - Possibility of overflow, inexactness of rounding
- **1 is multiplicative identity?** **YES**
- **Multiplication distributes over addtion?** **NO**
  - Possibility of overflow, inexactness of rounding

Montonicity

- $a \geq b\ \&\ c \geq\ 0 \Rightarrow a * c \geq b * c$**?** **ALMOST**
  - Except for infinities & NaNs

Floating Point in C

C Guarantees Two Levels

`float` **single precision**

`double` **double precision**

Conversions

- **Casting between `int`, `float`, and `double` changes numeric values**
- **`Double` or `float` to `int`**
  - Truncates fractional part
  - Like rounding toward zero
  - Not defined when out of range
    - » Generally saturates to TMin or TMax
- **`int` to `double`**
  - Exact conversion, as long as int has ≤ 53 bit word size
- **`int` to `float`**
  - Will round according to rounding mode

Answers to Floating Point Puzzles

```
int x = …;
float f = …;
double d = …;
```

Assume neither **d** nor **f** is NAN

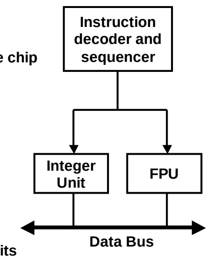
- `x == (int)(float) x` — **No: 24 bit significand**
- `x == (int)(double) x` — **Yes: 53 bit significand**
- `f == (float)(double) f` — **Yes: increases precision**
- `d == (float) d` — **No: loses precision**
- `f == -(-f);` — **Yes: Just change sign bit**
- `2/3 == 2/3.0` — **No: 2/3 == 0**
- `d < 0.0` ⇒ `((d*2) < 0.0)` — **Yes!**
- `d > f` ⇒ `-f < -d` — **Yes!**
- `d * d >= 0.0` — **Yes!**
- `(d+f)-d == f` — **No: Not associative**

IA32 Floating Point

History
- **8086: first computer to implement IEEE FP**
  - separate 8087 FPU (floating point unit)
- **486: merged FPU and Integer Unit onto one chip**

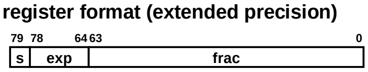
Summary
- **Hardware to add, multiply, and divide**
- **Floating point data registers**
- **Various control & status registers**

Floating Point Formats
- **single precision (C `float`): 32 bits**
- **double precision (C `double`): 64 bits**
- **extended precision (C `long double`): 80 bits**

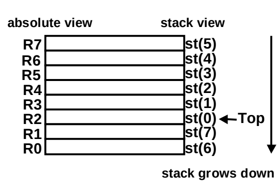


FPU Data Register Stack

FPU register format (extended precision)



FPU register stack
- **stack grows down**
  - wraps around from R0 -> R7
- **FPU registers are typically referenced relative to top of stack**
  - st(0) is top of stack (Top)
  - followed by st(1), st(2),…
- **push: increment Top, load**
- **pop: store, decrement Top**



FPU instructions

Large number of floating point instructions and formats
- **~50 basic instruction types**
- **load, store, add, multiply**
- **sin, cos, tan, arctan, and log!**

Sampling of instructions:

Instruction	Effect	Description
`fldz`	`push 0.0`	Load zero
`flds S`	`push S`	Load single precision real
`fmuls S`	`st(0) <- st(0)*S`	Multiply
`faddp`	`st(1) <- st(0)+st(1); pop`	Add and pop

Floating Point Code Example

Compute Inner Product of Two Vectors

- Single precision arithmetic
- Scientific computing and signal processing workhorse

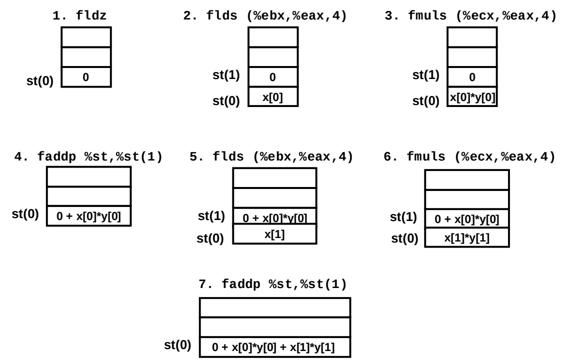
```
float ipf (float x[],
           float y[],
           int n)
{
  int i;
  float result = 0.0;

  for (i = 0; i < n; i++) {
    result += x[i] * y[i];
  }
  return result;
}
```

```
   pushl %ebp              # setup
   movl %esp,%ebp
   pushl %ebx

   movl 8(%ebp),%ebx       # %ebx=&x
   movl 12(%ebp),%ecx      # %ecx=&y
   movl 16(%ebp),%edx      # %edx=n
   fldz                    # push +0.0
   xorl %eax,%eax          # i=0
   cmpl %edx,%eax          # if i>=n done
   jge .L3
.L5:
   flds (%ebx,%eax,4)      # push x[i]
   fmuls (%ecx,%eax,4)     # st(0)*=y[i]
   faddp                   # st(1)+=st(0); pop
   incl %eax               # i++
   cmpl %edx,%eax          # if i<n repeat
   jl .L5
.L3:
   movl -4(%ebp),%ebx      # finish
   leave
   ret                     # st(0) = result
```


Inner product stack trace



Summary

IEEE Floating Point Has Clear Mathematical Properties

- Represents numbers of form $M \times 2^E$
- Can reason about operations independent of implementation
  - As if computed with perfect precision and then rounded
- Not the same as real arithmetic
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers

IA32 Floating Point is a Mess

- Ill-conceived, pseudo-stack architecture
- Covered in notes