Introduction to Computer Systems

15-213/18-243, fall 2009
3rd Lecture, Sep. 1st

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Last Time: Integers

- **Representation: unsigned and signed**
- **Conversion, casting**
  - Bit representation maintained but reinterpreted
- **Expanding, truncating**
  - Truncating = mod
- **Addition, negation, multiplication, shifting**
  - Operations are mod 2^w
- **Ordering properties do not hold**
  - u > 0 does not mean u + v > v
  - u, v > 0 does not mean u · v > 0

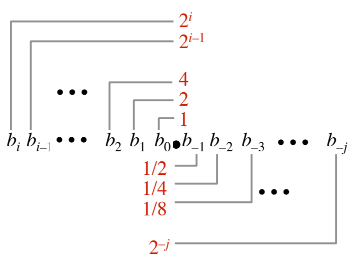
Today: Floating Point

- **Background: Fractional binary numbers**
- **IEEE floating point standard: Definition**
- **Example and properties**
- **Rounding, addition, multiplication**
- **Floating point in C**
- **Summary**

Fractional binary numbers

- **What is 1023.405_{10}?**
- **What is 1011.101_2?**

Fractional Binary Numbers



- **Representation**
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i} b_k \cdot 2^k$

Fractional Binary Numbers: Examples

Value	**Representation**
5-3/4	$\texttt{101.11}_2$
2-7/8	$\texttt{10.111}_2$
63/64	$\texttt{0.111111}_2$

- **Observations**
  - Divide by 2 by shifting right
  - Multiply by 2 by shifting left
  - Compare to shifting decimal numbers right or left
  - Numbers of form $\texttt{0.111111}\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Compare to $0.9999\ldots_{10} \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- **Limitation**
  - Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations
- **Value** **Representation**

Value	Representation
1/3	`0.0101010101[01]`$\ldots_2$
1/5	`0.001100110011[0011]`$\ldots_2$
1/10	`0.0001100110011[0011]`$\ldots_2$

- **Observation**
  - 0.1_{10} has no finite exact binary representation!

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IEEE Floating Point

- **IEEE Standard 754**
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs
- **Driven by numerical concerns**
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

- **Numerical Form:**

$$(-1)^s\ M\ \ 2^E$$

  - **Sign bit** s determines whether number is negative or positive
  - **Significand** M normally a fractional value in range [1.0,2.0).
  - **Exponent** E weights value by power of two
- **Encoding**
  - MSB `s` is sign bit s
  - `exp` field encodes E (but is not equal to E)
  - `frac` field encodes M (but is not equal to M)

s	exp	frac

Precisions

- **Single precision: 32 bits**

s	exp	frac
1	8	23

- **Double precision: 64 bits**

s	exp	frac
1	11	52

- **Extended precision: 80 bits (Intel only)**

s	exp	frac
1	15	63 or 64

Normalized Values

- **Condition:** `exp` $\neq$ `000…0` **and** `exp` $\neq$ `111…1`
- **Exponent coded as *biased* value:** $E = Exp - Bias$
  - *Exp*: unsigned value `exp`
  - *Bias* = $2^{e-1} - 1$, where e is number of exponent bits
    - Single precision: 127 (*Exp*: 1…254, *E*: -126…127)
    - Double precision: 1023 (*Exp*: 1…2046, *E*: -1022…1023)
- **Significand coded with implied leading 1:** M = `1.xxx…x`$_2$
  - `xxx…x`: bits of `frac`
  - Minimum when `000…0` (M = 1.0)
  - Maximum when `111…1` ($M = 2.0 - \varepsilon$)
  - Why does M range from 1 to 2-? Why not 0 to 1-?
  - Get extra leading bit for "free"

Normalized Encoding Example

- Value: `Float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$
- Significand
  M = 1.$\underline{1101101101101}_2$
  `frac`= $\underline{1101101101101}0000000000_2$
- Exponent
  E = 13
  $Bias$ = 127
  Exp = 140 = 10001100_2
- Result:

s	exp	frac
0	10001100	11011011011010000000000

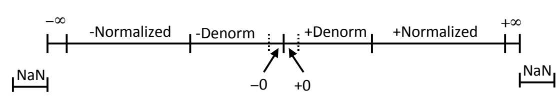
Denormalized Values

- Condition: `exp` = 000…0
- Exponent value: E = 1 – $Bias$ (instead of E = 0 – Bias)
- Significand coded with implied leading 0: M = 0.`xxx…x`$_2$
  - `xxx…x`: bits of `frac`
- Cases
  - `exp` = 000…0, `frac` = 000…0
    - Represents value 0
    - Note distinct values: +0 and –0 (why?)
  - `exp` = 000…0, `frac` ≠ 000…0
    - Numbers very close to 0.0
    - Lose precision as get smaller
    - Equispaced

Special Values

- Condition: `exp` = 111…1
- Case: `exp` = 111…1, `frac` = 000…0
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞
- Case: `exp` = 111…1, `frac` ≠ 000…0
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ − ∞, ∞ ∗ 0

Visualization: Floating Point Encodings



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Tiny Floating Point Example

s	exp	frac
1	4	3

- 8-bit Floating Point Representation
  - the sign bit is in the most significant bit.
  - the next four bits are the exponent, with a bias of 7.
  - the last three bits are the `frac`
- Same general form as IEEE Format
  - normalized, denormalized
  - representation of 0, NaN, infinity

Is 8-bit Float Just an Example?

s	exp	frac
1	3	4

- uLaw Audio Representation
  - An 8-bit float used for digital telephony in North America/Japan
- We'll hear some examples later
- Small floats also used in GPUs!

Dynamic Range (Positive Only)

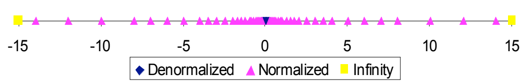
	s exp frac	E	Value	
Denormalized numbers	0 0000 000	-6	0	
	0 0000 001	-6	1/8*1/64 = 1/512	closest to zero
	0 0000 010	-6	2/8*1/64 = 2/512	
	…			
	0 0000 110	-6	6/8*1/64 = 6/512	
	0 0000 111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0 0001 000	-6	8/8*1/64 = 8/512	smallest norm
	0 0001 001	-6	9/8*1/64 = 9/512	
	…			
	0 0110 110	-1	14/8*1/2 = 14/16	
	0 0110 111	-1	15/8*1/2 = 15/16	closest to 1 below
	0 0111 000	0	8/8*1 = 1	
	0 0111 001	0	9/8*1 = 9/8	closest to 1 above
	0 0111 010	0	10/8*1 = 10/8	
	…			
	0 1110 110	7	14/8*128 = 224	
	0 1110 111	7	15/8*128 = 240	largest norm
	0 1111 000	n/a	inf	

Distribution of Values

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

s	exp	frac
1	3	2

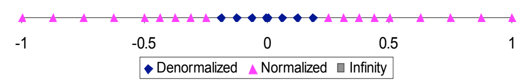
- Notice how the distribution gets denser toward zero.



Distribution of Values (close-up view)

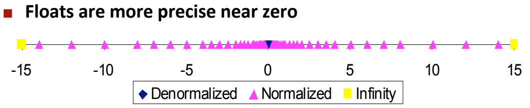
- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3

s	exp	frac
1	3	2

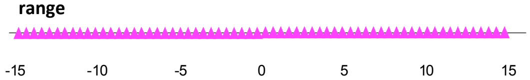


Sound Examples

- Floats are more precise near zero



- Fixed-point numbers quantize uniformly throughout their range



Interesting Numbers {single,double}

Description	exp	frac	Numeric Value
Zero	00…00	00…00	0.0
Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
Single ≈ 1.4×10^{-45}			
Double ≈ 4.9×10^{-324}			
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
Single ≈ 1.18×10^{-38}			
Double ≈ 2.2×10^{-308}			
Smallest Pos. Normalized		00…01	00…00 $1.0 \times 2^{-\{126,1022\}}$
Just larger than largest denormalized			
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
Single ≈ 3.4×10^{38}			
Double ≈ 1.8×10^{308}			

Special Properties of Encoding

- **FP Zero Same as Integer Zero**
  - All bits = 0
- **Can (Almost) Use Unsigned Integer Comparison**
  - Must first compare sign bits
  - Must consider -0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

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Floating Point Operations: Basic Idea

- `x +f y = Round(x + y)`
- `x xf y = Round(x x y)`
- **Basic idea**
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into **`frac`**

Rounding

- **Rounding Modes (illustrate with $ rounding)**

	$1.40	$1.60	$1.50	$2.50	–$1.50
Towards zero	$1	$1	$1	$2	–$1
Round down (-∞)	$1	$1	$1	$2	–$2
Round up (+∞)	$2	$2	$2	$3	–$1
Nearest Even (default)	$1	$2	$2	$2	–$2

- **What are the advantages of the modes?**

Closer Look at Round-To-Even

- **Default Rounding Mode**
  - Hard to get any other kind without dropping into assembly
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated
- **Applying to Other Decimal Places / Bit Positions**
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

1.2349999	1.23	(Less than half way)
1.2350001	1.24	(Greater than half way)
1.2350000	1.24	(Half way—round up)
1.2450000	1.24	(Half way—round down)

Rounding Binary Numbers

- **Binary Fractional Numbers**
  - "Even" when least significant bit is `0`
  - "Half way" when bits to right of rounding position = $100\ldots_2$
- **Examples**
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

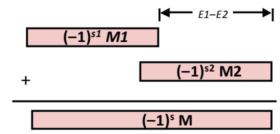
$(-1)^{s1}\ M1\ \ 2^{E1}\ \ \times\ \ (-1)^{s2}\ M2\ \ 2^{E2}$

- **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - Sign s: $s1$ ^ $s2$
  - Significand M: $M1 * M2$
  - Exponent E: $E1 + E2$
- **Fixing**
  - If $M \geq 2$, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit `frac` precision
- **Implementation**
  - Biggest chore is multiplying significands

Floating Point Addition

$(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$

Assume $E1 > E2$



- **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: $E1$
- **Fixing**
  - If $M \geq 2$, shift M right, increment E
  - if $M < 1$, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit `frac` precision

Mathematical Properties of FP Add

- **Compare to those of Abelian Group**
  - Closed under addition? ***Yes***
    - But may generate infinity or NaN
  - Commutative? ***Yes***
  - Associative? ***No***
    - Overflow and inexactness of rounding
  - 0 is additive identity? ***Yes***
  - Every element has additive inverse ***Almost***
    - Except for infinities & NaNs
- **Monotonicity**
  - $a \geq b \Rightarrow a+c \geq b+c$? ***Almost***
    - Except for infinities & NaNs

Mathematical Properties of FP Mult

- **Compare to Commutative Ring**
  - Closed under multiplication? ***Yes***
    - But may generate infinity or NaN
  - Multiplication Commutative? ***Yes***
  - Multiplication is Associative? ***No***
    - Possibility of overflow, inexactness of rounding
  - 1 is multiplicative identity? ***Yes***
  - Multiplication distributes over addition? ***No***
    - Possibility of overflow, inexactness of rounding
- **Monotonicity**
  - $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? ***Almost***
    - Except for infinities & NaNs

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Floating Point in C

- **C Guarantees Two Levels**
  - `float` single precision
  - `double` double precision
- **Conversions/Casting**
  - Casting between `int`, `float`, and `double` changes bit representation
  - `Double`/`float` → `int`
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - `int` → `double`
    - Exact conversion, as long as int has ≤ 53 bit word size
  - `int` → `float`
    - Will round according to rounding mode

Floating Point Puzzles

- For each of the following C expressions, either:
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither
d nor f is NaN

- `x == (int)(float) x`
- `x == (int)(double) x`
- `f == (float)(double) f`
- `d == (float) d`
- `f == -(-f);`
- `2/3 == 2/3.0`
- `d < 0.0` ⇒ `((d*2) < 0.0)`
- `d > f` ⇒ `-f > -d`
- `d * d >= 0.0`
- `(d+f)-d == f`

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Summary

- IEEE Floating Point has clear mathematical properties
- Represents numbers of form $M \times 2^E$
- One can reason about operations independent of implementation
  - As if computed with perfect precision and then rounded
- Not the same as real arithmetic
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers

More Slides

Creating Floating Point Number

- Steps
  - Normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding

7	6 … 3	2 … 0
s	exp	frac

- Case Study
  - Convert 8-bit unsigned numbers to tiny floating point format
  - Example Numbers

128	10000000
15	00001101
33	00010001
35	00010011
138	10001010
63	00111111

Normalize

7	6 … 3	2 … 0
s	exp	frac

- Requirement
  - Set binary point so that numbers of form 1.xxxxx
  - Adjust all to have leading one
    - Decrement exponent as shift left

Value	*Binary*	*Fraction*	*Exponent*
128	10000000	1.0000000	7
15	00001101	1.1010000	3
17	00010001	1.0001000	5
19	00010011	1.0011000	5
138	10001010	1.0001010	7
63	00111111	1.1111100	5

Rounding

1.BBGRXXX

Guard bit: LSB of result

Round bit: 1st bit removed

Sticky bit: OR of remaining bits

- **Round up conditions**
  - Round = 1, Sticky = 1 ➔ > 0.5
  - Guard = 1, Round = 1, Sticky = 0 ➔ Round to even

Value	*Fraction*	*GRS*	*Incr?*	*Rounded*
128	1.0000000	00*	N	1.000
15	1.1010000	10*	N	1.101
17	1.0001000	010	N	1.000
19	1.0011000	110	Y	1.010
138	1.0001010	011	Y	1.001
63	1.1111100	111	Y	10.000

Postnormalize

- **Issue**
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

Value	*Rounded*	*Exp*	*Adjusted*	*Result*
128	1.000	7		128
15	1.101	3		15
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	1.000/6	64