# Johnson's Algorithm:

Solves the all pairs shortest path (APSP) problem. Works with negative weight edges (but not if there are negative cycles)

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Could solve APSP problem by running Bellman Ford from each vertex

$$\text{Work} = O(mn) \times n = O(mn^2)$$

This is not efficient

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Idea: Two Phases

1) ~~Bellman~~ Run Bellman Ford and use result to convert graph to have no negative weight edges
2) Run Dijkstra from each vertex

| | Work | Span |
|---|---|---|
| Bellman Ford | $O(mn)$ | $O(n \log n)$ |
| Dijkstra $\times n$ | $O(m \log n) \times n$ | $O(m \log n)$ |
| Total | $O(mn \log n)$ | $O(m \log n)$ |

Observation:

Assign a "potential" $p(v)$ to each vertex $v \in V$.

Then ~~Then~~ adjust edges:

$$w'(u,v) = w(u,v) + p(u) - p(v)$$

least weight path

**Claim:**

$$\delta_G(u,v) = \delta_{G'}(u,v) - p(u) + p(v)$$

original weights

adjusted weights

"Proof" by picture:

$$a \xrightarrow[-p(b)]{+p(a)} b \xrightarrow[-p(c)]{+p(b)} c \xrightarrow[-p(d)]{+p(c)} d$$

$$\delta_{G'}(a,d) = w(a,b) + p(a) - \cancel{p(b)} + w(b,c) + \cancel{p(b)} - \cancel{p(c)}$$

by cancellation

$$+ w(c,d) + \cancel{u(c)} - u(d)$$

$$= w(a,b) + w(b,c) + w(c,d) + p(a) - p(d)$$

$$= \delta_G(a,d) + p(a) - p(d)$$

therefore by flipping sides

$$\delta_G(a,d) = \delta_{G'}(a,d) - p(a) + p(d)$$

# Johnsons algorithm

1) Add a dummy vertex $s$ to $G$ with edge to every $v \in V$ with weight $0$.
(Note: instead could use any $s \in V$ that can reach all other vertices)

2) Run Bellman Ford on new Graph ~~starting~~ with ~~a~~ source $s$.

3) use $\delta(s,v) = p(v)$ to adjust weights

4) Run Dijkstra to find all $\delta_{G'}(u,v)$, return $\boxed{\delta_G(u,v) = \delta_{G'}(u,v) - d(u) + d(v)}$

Claim: no negative weight edges in

$$w'(u,v) = w(u,v) + \underbrace{p(u)}_{\delta(s,u)} - \underbrace{p(v)}_{\delta(s,v)}$$

Why: General property of shortest paths

$$\delta(s,v) \leq \delta(s,u) + w(u,v)$$

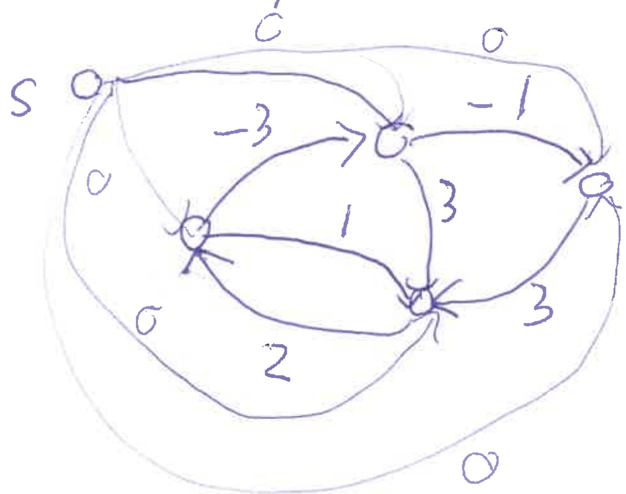
Since ~~the~~ shortest path cannot be larger than the shortest path through $u$.

Hence

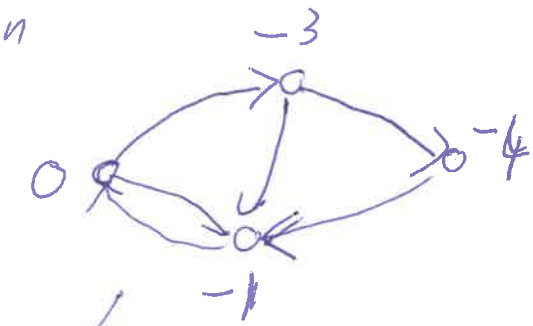
$$\begin{aligned} 0 &\leq \delta(s,u) + w(u,v) - \delta(s,v) \\ &= w(u,v) + \delta(s,u) - \delta(s,v) \end{aligned}$$

Q.E.D.

Example:

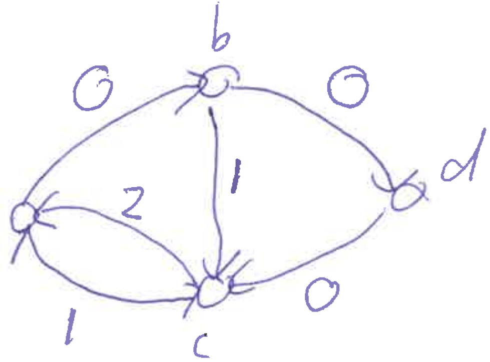



Bellman Ford $\longrightarrow$




adjusted weights

$G' =$




No negative weights

$0 - 0 + (-4) = -4$

e.g.

$\delta_{G'}(a,d) = 0$ $\qquad$ $\delta_G(a,d) = \cancel{0 - 4 + 0 = -4}$

$\delta_{G'}(b,a) = \cancel{0}\ \underline{1}$ $\qquad$ $\delta_G(b,a) = 1 - (-3) + 0 = 4$