Recitation 3

Scan

3.1 Announcements

- *SkylineLab* has been released, and is due **Friday afternoon**. It's worth 125 points.
- *BignumLab* will be released on Friday.

3.2 What is `scan`?

In the `SEQUENCE` library, there is a symmetry among certain aggregation functions:

Sequential	**Parallel**
`iterate`	`reduce`
`iteratePrefixes`	`scan`
`iteratePrefixesIncl`	`scanIncl`

We can see this symmetry in their types...

`iterate`	$(\beta * \alpha \to \beta) \to \beta \to \alpha\ \texttt{seq} \to \beta$
`reduce`	$(\alpha * \alpha \to \alpha) \to \alpha \to \alpha\ \texttt{seq} \to \alpha$
`iteratePrefixes`	$(\beta * \alpha \to \beta) \to \beta \to \alpha\ \texttt{seq} \to \beta\ \texttt{seq} * \beta$
`scan`	$(\alpha * \alpha \to \alpha) \to \alpha \to \alpha\ \texttt{seq} \to \alpha\ \texttt{seq} * \alpha$
`iteratePrefixesIncl`	$(\beta * \alpha \to \beta) \to \beta \to \alpha\ \texttt{seq} \to \beta\ \texttt{seq}$
`scanIncl`	$(\alpha * \alpha \to \alpha) \to \alpha \to \alpha\ \texttt{seq} \to \alpha\ \texttt{seq}$

...as well as their output behavior: each of the parallel functions has output identical to its sequential analog under the condition that the first two arguments are an *associative function* and a corresponding *identity*, respectively.

Definition 3.1. *A function f is associative if for every x, y, z,*

$$f(f(x, y), z) = f(x, f(y, z)).$$

Definition 3.2. *A value b is an identity of a binary function f if for every x,*

$$f(b, x) = x = f(x, b).$$

So, for now, you can think of `scan` as a magical function which performs iteration of an associative function in parallel. If the function is constant-time, then an application of scan has linear work and logarithmic span.

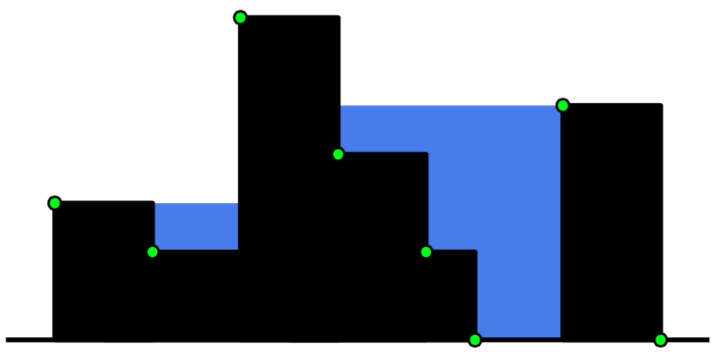
Remark 3.3. *In reality, we can relax the constraint on the identity. It only needs to be an identity for the values encountered during the execution of the `reduce`, `scan`, or `scanIncl`. For example, if S is a sequence of non-negative integers, then (`scan Int.max 0` S) will still be logically equivalent to (`iteratePrefixes Int.max 0` S), despite the fact that, in general, 0 is not an identity for `Int.max`.*

3.3 Skyline-Fill

For this example, we'll use the same conventions given in *SkylineLab*:

- Skylines are sequences of points (x, y) sorted by x-coordinate,
- all x-coordinates are unique and non-negative, and
- all y-coordinates (heights) are non-negative.

Imagine pouring water on a skyline. How much water can it hold?



Task 3.4. *Implement the function*

$$\texttt{val fill : (int * int) Seq.t} \rightarrow \texttt{int}$$

where (`fill` S) *returns the area of water which can fill the skyline* S. *Your implementation should have* $O(|S|)$ *work and* $O(\log |S|)$ *span.*

3.4 A Group at Dinner

A group of n friends sit around a circular table at a restaurant. Some of them know what they want to order; some of them don't. The ones who don't know what to order decide to pick the same thing as the person on their left.

Task 3.5. *Implement the function*

$$\texttt{val groupOrder : (int} \rightarrow \alpha \texttt{ option)} \rightarrow \texttt{int} \rightarrow \alpha \texttt{ Seq.t}$$

where (`groupOrder` f n) *returns the sequence of orders of a group of n people. $f(i)$ is either the preferred order of the i^{th} person, or* `NONE` *if they don't know what they want. Assume the people are labeled 0 to $n-1$ counter-clockwise, and that at least one person originally knows what they want to order. Your implementation should have $O(n)$ work and $O(\log n)$ span.*

3.5 Bonus Exercises

Exercise 3.6. *Implement* `parenMatch` *(from the previous recitation) using* `scan` *such that it has linear work and logarithmic span. Try adapting the iterative approach.*

Exercise 3.7. *Implement* `parenDist` *(from* ParenLab*) using* `scan` *such that it has linear work and logarithmic span.*

Exercise 3.8. *Did you know that you can calculate the first n Fibonacci numbers in $O(n)$ time and $O(\log n)$ span? We claim that if we extend the Fibonacci sequence as so...*

$$\begin{aligned} F_{-1} &= 1 \\ F_0 &= 0 \\ F_1 &= 1 \\ &\vdots \\ F_n &= F_{n-1} + F_{n-2} \end{aligned}$$

...that the following holds for $n \geq 0$ (easily provable via induction):

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

Using this fact, implement a function

```
val fibs : int → int Seq.t
```

which returns the first n Fibonacci numbers in $O(n)$ work and $O(\log n)$ span. Use `scan` *to compute prefixes of matrix multiplications.*

Exercise 3.9. *Implement a function*

```
val parenPairs : Paren.t Seq.t → (int * int) Seq.t
```

where (`parenPairs` *S) returns a sequence of index-pairs where each pair contains the index of a parenthesis as well as its matching partner. For example the input* `(())()` *should yield some permutation of $\langle (0,3),(4,5),(1,2)\rangle$. Your implementation should have $O(|S|\log|S|)$ work and $O(\log^2|S|)$ span.*
Hint: try marking each parenthesis with how many other parentheses are enclosing it. You might also need to `sort` *at some point...*

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