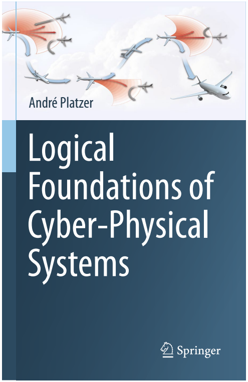
Stability & Switched Systems

Logical Foundations of Cyber-Physical Systems



Stefan Mitsch

Carnegie Mellon University
Computer Science Department

Outline

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1. Beyond Safety
2. Stability
3. Switched Systems
4. Summary

What Else Could Possibly Go Wrong?

Proposition (System Proved Safe)

$$v \le v_c + \delta \to [(ctrl; ode)^*] v \le v_c + \delta$$

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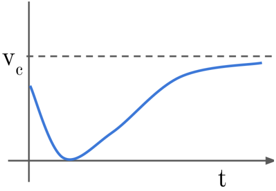

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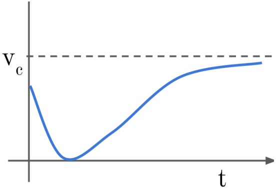

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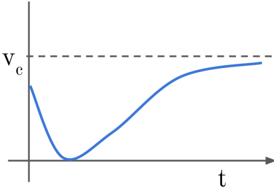

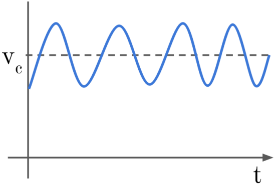

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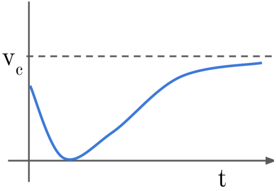

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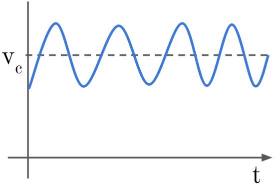

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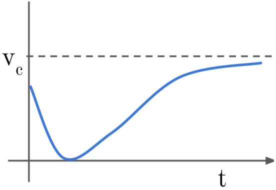

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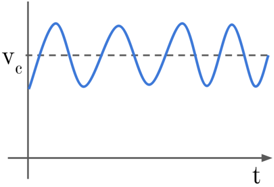

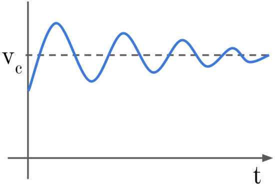

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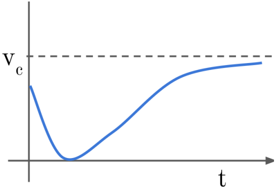

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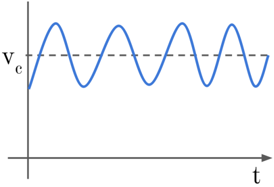

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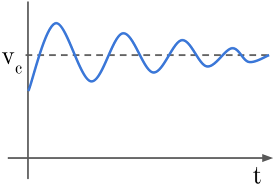

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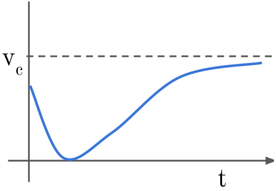

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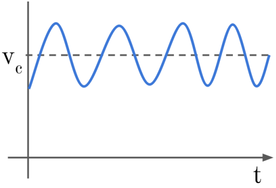

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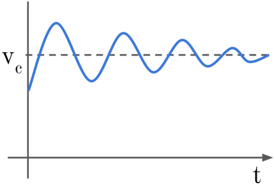
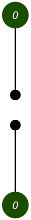
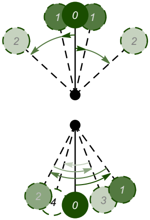


Theory



Practice



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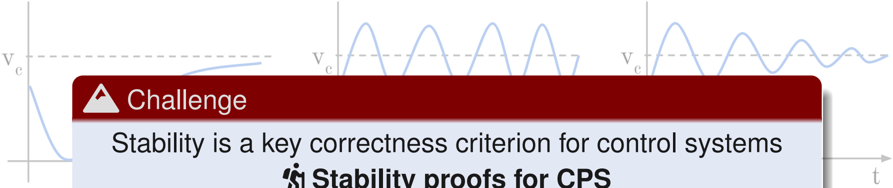


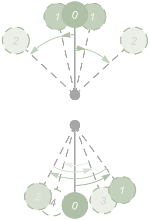
Challenge

Stability is a key correctness criterion for control systems

Stability proofs for CPS

Theory

Practice



Outline

Asymptotic Stability in dL

- **Stability** stay close to origin when slightly perturbed
- **Attractivity** dissipate energy when slightly perturbed

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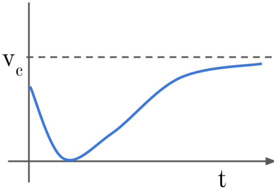

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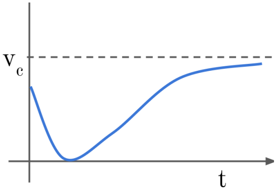

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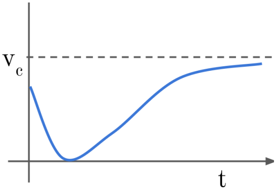

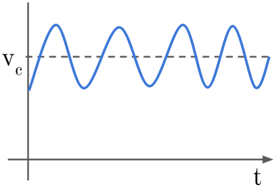

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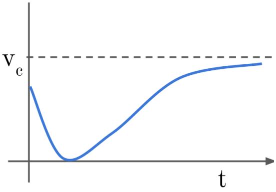

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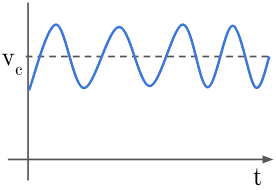

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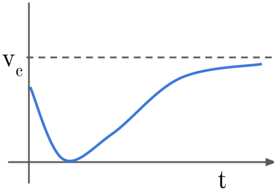

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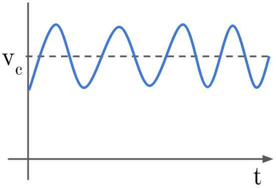

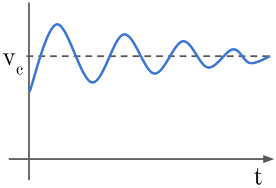

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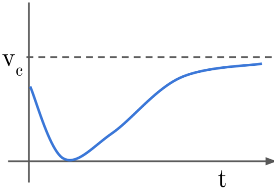

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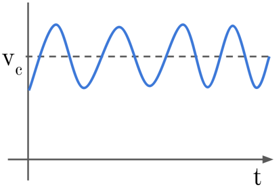

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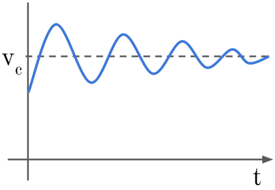

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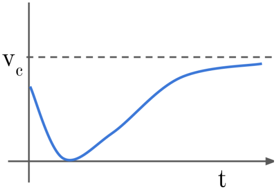

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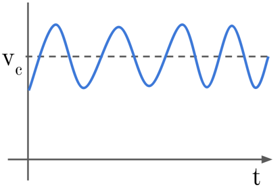

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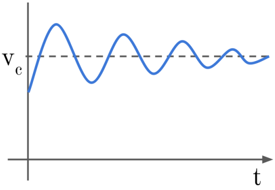
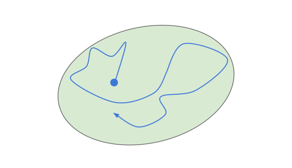


Safety and Liveness

Stability: $[\alpha]P$



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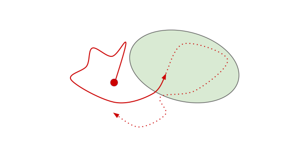

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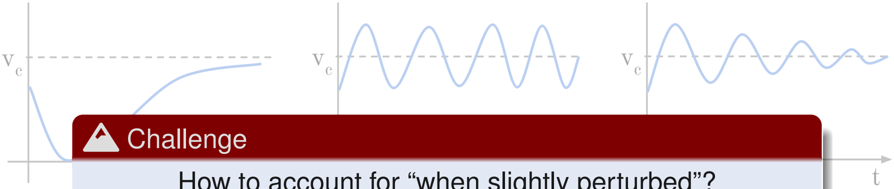

Challenge

How to account for “when slightly perturbed”?

Safety and Liveness

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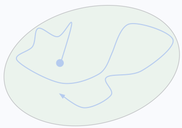

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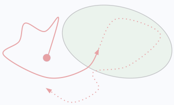

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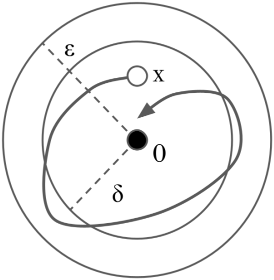

Origin 0 of an ODE $x' = f(x)$ with solution $x(t) : [0, T) \to \mathbb{R}^n$ is stable if

- for all $\varepsilon > 0$
- there exists $\delta > 0$
- s.t. for all $\|x(0)\| < \delta$
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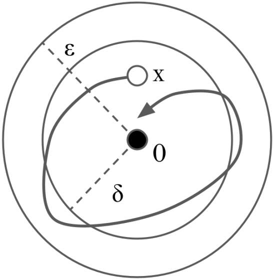
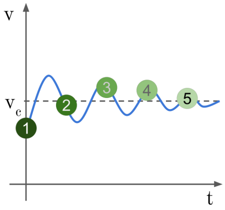


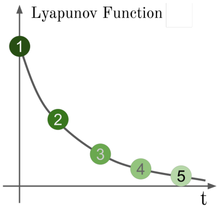
Stability

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \, (\|x\| < \delta \to [x' = f(x)] \|x\| < \varepsilon)$$

Lyapunov-functions V are energy-like functions to certify asymptotic stability

- V, V' are continuous
- $V(0) = 0$
- $V(x) > 0$ for all $\|x\| > 0$
- $V' \leq 0$





Lemma (Lyapunov Proof Rule)

Rule $\text{Lyap}_{\geq}$ is derivable in dL

$$\text{Lyap}_{\geq}\ \frac{\vdash f(0) = 0 \land V(0) = 0 \quad 0 < \|x\|^2 \vdash V > 0 \land V' \leq 0}{\vdash \forall \varepsilon{>}0\, \exists \delta{>}0\, \forall x\, (\|x\|^2 < \delta^2 \rightarrow [x' = f(x)]\|x\|^2 < \varepsilon^2)}$$

Attractivity in dL

- **Attractivity** dissipate energy when slightly perturbed

Origin 0 of an ODE $x' = f(x)$ with solution $x(t) : [0, T) \to \mathbb{R}^n$ is attractive if

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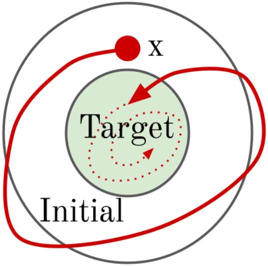

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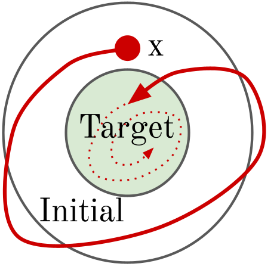

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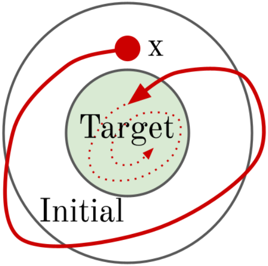

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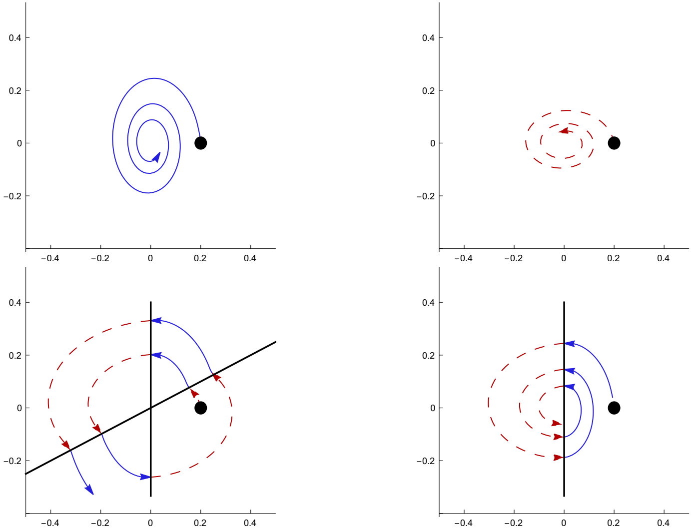
Pre-attractivity

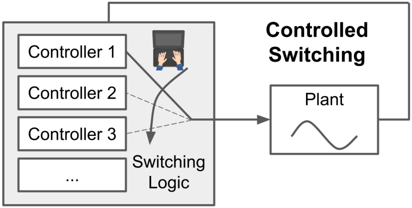
$$\forall \varepsilon > 0 \, \forall \delta > 0 \, \exists T \geq 0 \, \forall x \, (\|x\| < \delta \to [t := 0; \{x' = f(x), t' = 1\}](t \geq T \to \|x\| < \varepsilon))$$

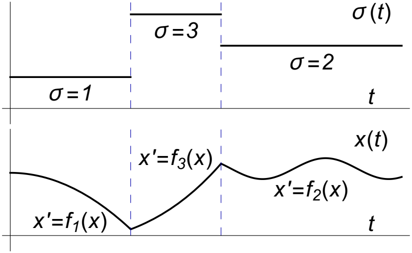
Outline

Switching between stable ODEs can be unstable



- Family $\mathscr{P}$ of ODEs $x' = f_p(x), p \in \mathscr{P}$
- Switching signal $\sigma(t)$ chooses ODE





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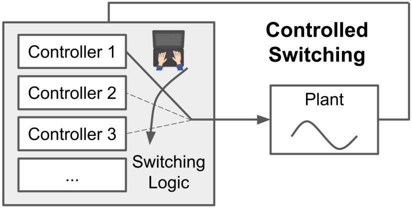

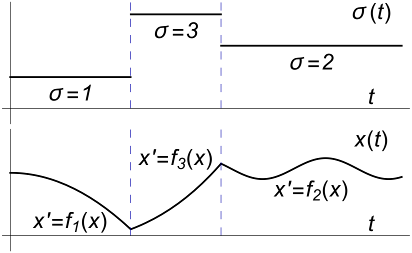

Arbitrary Switching as Hybrid Program

Non-deterministic choice between all ODEs: $\left(\bigcup_{p\in\mathscr{P}} x' = f_p(x)\right)^*$

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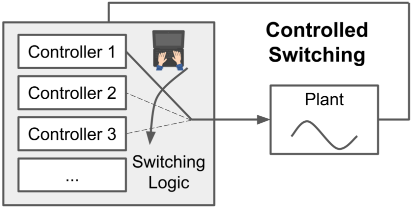

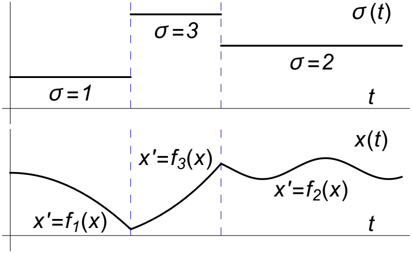

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Controlled Switching as Hybrid Program

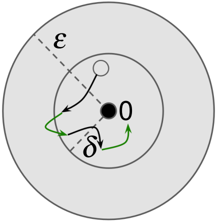
Controlled choice between ODEs: $(p := \mathrm{ctrl}(x); x' = f_p(x))^*$

Construct loop invariants that imply stability

Arbitrary Switching: Common Lyapunov Function

$$\alpha_{\text{arb}} \equiv \left(\bigcup_{p\in\mathscr{P}} x' = f_p(x)\right)^*$$

$$\forall\varepsilon{>}0\,\exists\delta{>}0\,\forall x\,(\|x\|{<}\delta \to [\alpha_{\text{arb}}]\|x\|{<}\varepsilon)$$



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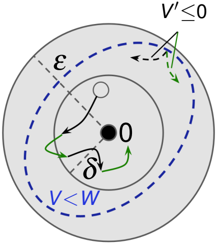

$$\alpha_{\text{arb}} \equiv \left(\bigcup_{p\in\mathscr{P}} x' = f_p(x)\right)^*$$

$$\forall \varepsilon{>}0\, \exists \delta{>}0\, \forall x\, (\|x\|{<}\delta \rightarrow [\alpha_{\text{arb}}]\|x\|{<}\varepsilon)$$

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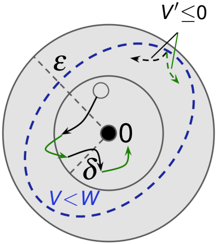
$$\alpha_{\text{arb}} \equiv \left(\bigcup_{p\in\mathscr{P}} x' = f_p(x)\right)^*$$

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \, (\|x\| < \delta \rightarrow [\alpha_{\text{arb}}] \|x\| < \varepsilon)$$



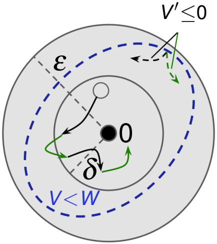
$Inv \equiv \|x\| < \varepsilon \land V < W$

Construct loop invariants that imply stability

Arbitrary Switching: Common Lyapunov Function

$$\alpha_{\text{arb}} \equiv \left(\bigcup_{p\in\mathscr{P}} x' = f_p(x)\right)^*$$

$$\forall\varepsilon{>}0\,\exists\delta{>}0\,\forall x\,(\|x\|{<}\delta \to [\alpha_{\text{arb}}]\|x\|{<}\varepsilon)$$



$$Inv \equiv \|x\| < \varepsilon \land \bigvee_{p\in\mathscr{P}} (u{=}p \land V_p{<}W)$$

Controlled Switching: Multiple Lyapunov Functions

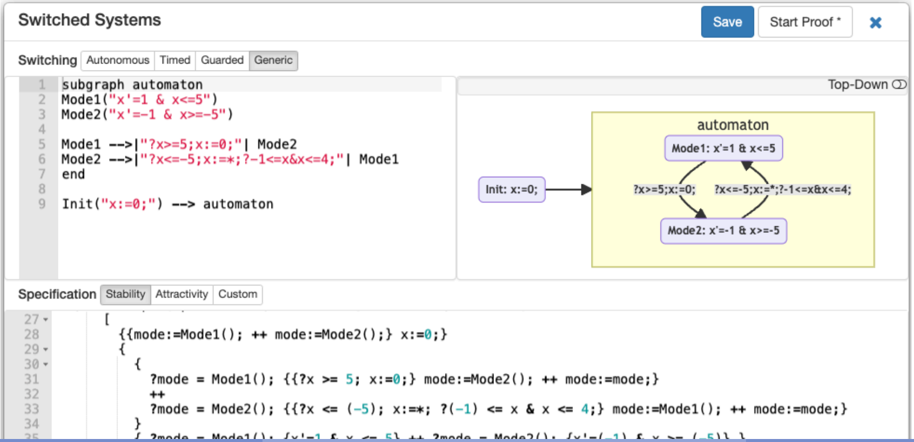
$$\alpha_{\text{ctrl}} \equiv (u := \text{ctrl}(x); x' = f_u(x))^*$$

$$\forall\varepsilon{>}0\,\exists\delta{>}0\,\forall x\,(\|x\|{<}\delta \to [\alpha_{\text{ctrl}}]\|x\|{<}\varepsilon)$$

Outline

Stability is a key correctness criterion for real-world safety

- **Stability** stay close to origin when slightly perturbed
- **Attractivity** dissipate energy when slightly perturbed
- **Lyapunov functions** certify stability
- **Switching** needs care to not cause instability



Yong Kiam Tan and André Platzer.
Deductive stability proofs for ordinary differential equations.
In *Tools and Algorithms for the Construction and Analysis of Systems - 27th International Conference, TACAS 2021, Luxembourg City, Luxembourg, March 27 - April 1, 2021, Proceedings, Part II*, pages 181–199, 2021.
doi:10.1007/978-3-030-72013-1_10.

Yong Kiam Tan and André Platzer.
Switched systems as hybrid programs.
In *7th IFAC Conference on Analysis and Design of Hybrid Systems, ADHS 2021, Brussels, Belgium, July 7-9, 2021*, pages 247–252, 2021.
doi:10.1016/j.ifacol.2021.08.506.

Yong Kiam Tan, Stefan Mitsch, and André Platzer.
Verifying switched system stability with logic.
In *HSCC '22: 25th ACM International Conference on Hybrid Systems: Computation and Control, Milan, Italy, May 4 - 6, 2022*, pages 2:1–2:11, 2022.
doi:10.1145/3501710.3519541.