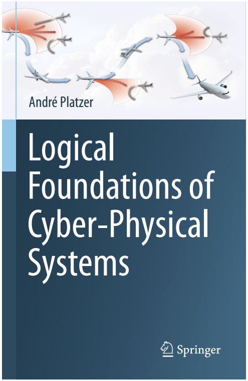
# 09: Reactions & Delays

## Logical Foundations of Cyber-Physical Systems

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Carnegie Mellon University
Computer Science Department

# Outline

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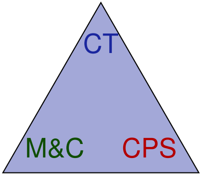
using loop invariants
design time-triggered control
design-by-invariant




modeling CPS
designing controls
time-triggered control
reaction delays
discrete sensing

semantics of time-triggered control
operational effect
finding control constraints
model-predictive control

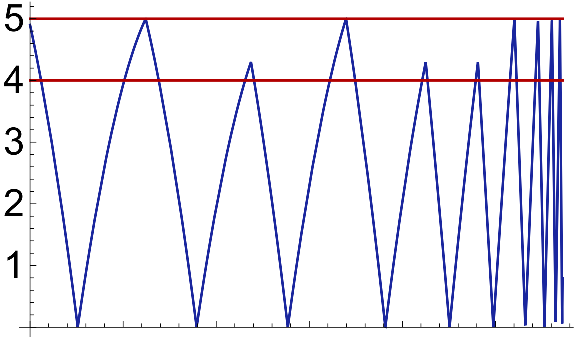
# Outline

# Quantum's Ping-Pong Proof Invariants

**Proposition (Quantum can play ping-pong safely)**

$$0 \le x \land x \le 5 \land v \le 0 \land g > 0 \land 1 \ge c \ge 0 \land f \ge 0 \to$$
$$\big[\big(\big(\{x' = v, v' = -g \,\&\, x \ge 0 \land x{\le}5\} \cup \{x' = v, v' = -g \,\&\, x{\ge}5\}\big);$$
$$\text{if}(x{=}0)\; v := -cv \;\text{else if}(4{\le}x{\le}5 \land v{\ge}0)\; v := -fv\big)^*\big](0{\le}x{\le}5)$$

**Proof** **@invariant**$(0{\le}x{\le}5 \land (x = 5 \to v{\le}0))$



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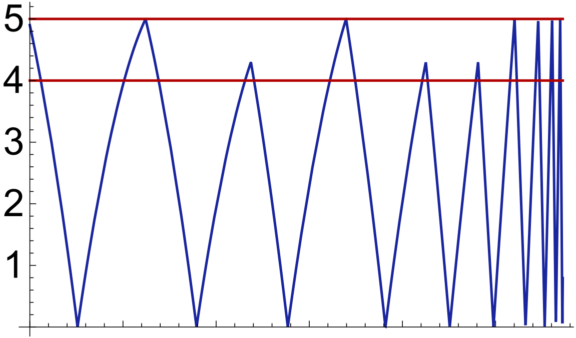
**Proof** **@invariant**$(0 \le x \le 5 \wedge (x = 5 \rightarrow v \le 0))$




Just can't implement . . .

# Physical vs. Controller Events

## Physical vs. Controller Events

1. Justifiable: Physical events (on ground $x = 0$)
2. Justifiable: Physical evolution domains (above ground $x \geq 0$)
3. Questionable: Controller evolution domain ($x \leq 5$)
4. Unlike physics, controllers won't run *all* the time. Just fairly often.
5. Controllers cannot sense and compute all the time.

If you expect the world to change for your controller's sake, you may be in for a surprise.

**Conjecture (Quantum can play ping-pong safely)**

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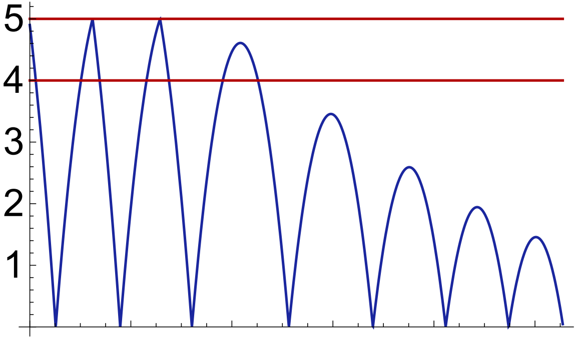
**Proof?** Ask René Descartes



# Back to the Drawing Desk: Quantum the Ping-Pong Ball

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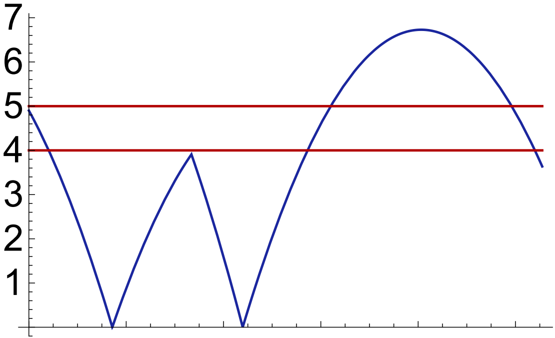
**Proof?** Ask René Descartes who says no!

Could miss if-then event



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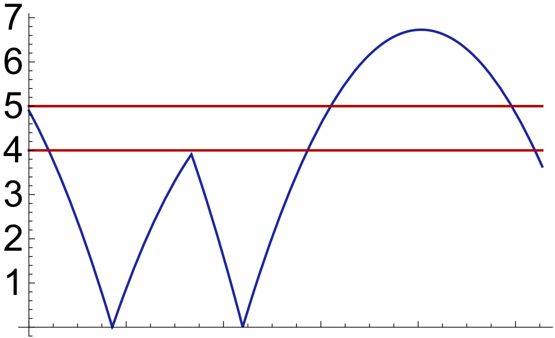

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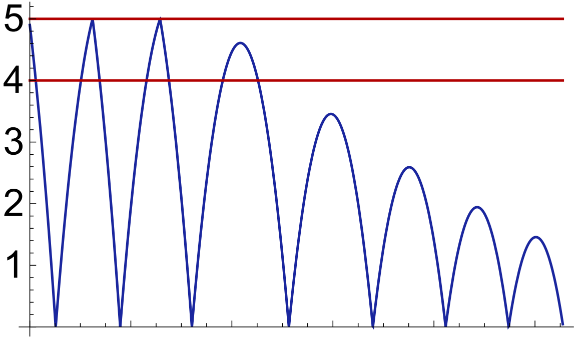

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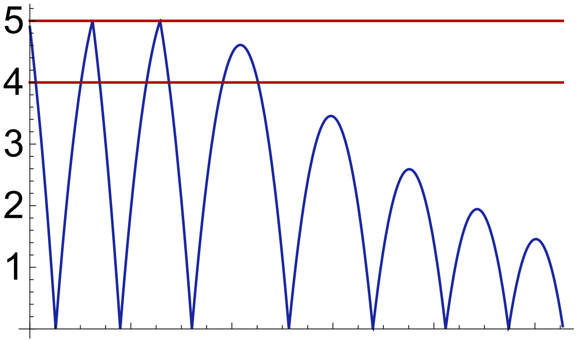


Wind up a clock

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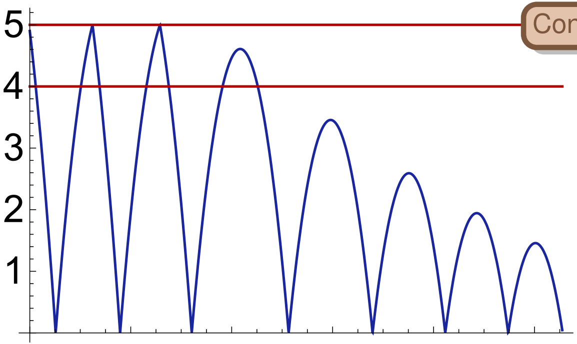

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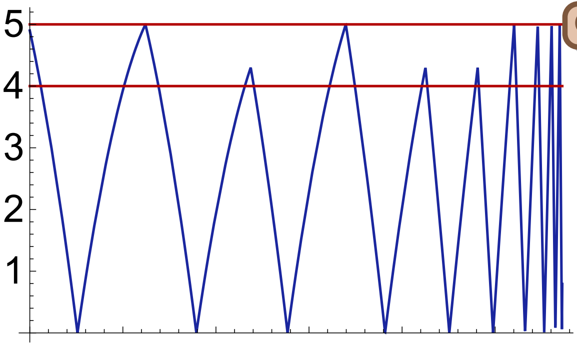
**Proof?** Ask René Descartes

Could act early or late



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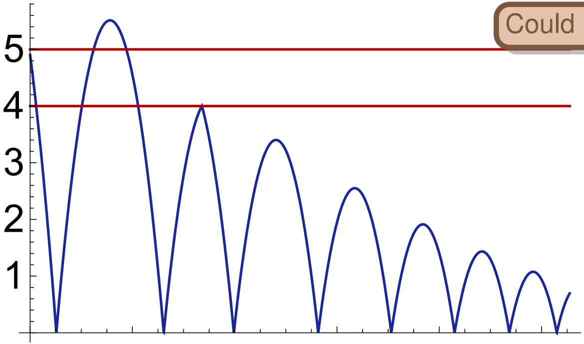
**Proof?** Ask René Descartes who says no!

Could miss event off control cycle



# Sampling May Miss Events

## Sampling vs. Events

1. Periodically/frequently monitor for an event with a polling frequency / reaction time
2. Sampling may make the controller miss events
   - Indicates discrepancy between event-triggered idea vs. time-triggered implementation
   - Indicates poor event abstraction
   - Consequence: event-triggered design would likely experience problems (unsafety) at runtime
3. Challenging: when slow controllers monitor small regions of a fast moving system
4. Controller needs to be aware of its own sampling interval to predict ahead

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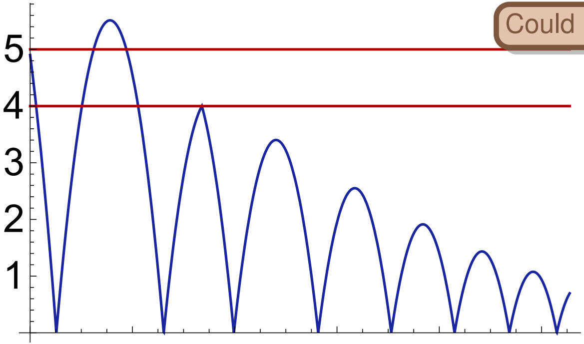

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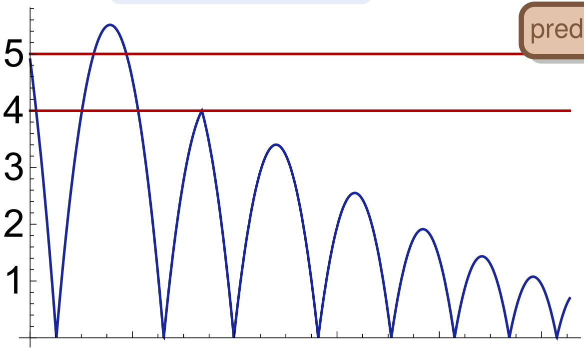
**Proof?** Ask René Descartes

predict t=1s: $x + vt - \frac{g}{2}t^2 > 5$



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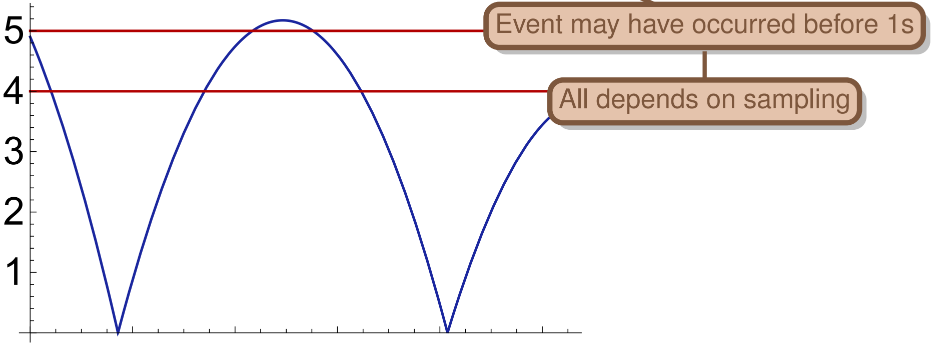

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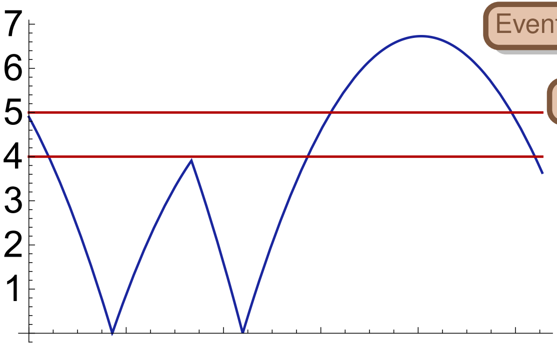
Event may have occurred before 1s

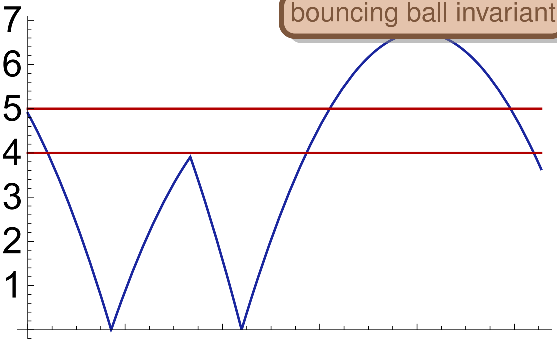
All depends on sampling



## Design-by-Invariant

$2gx = 2gH - v^2 \land x \geq 0 \land c = 1 \land g > 0$

bouncing ball invariant



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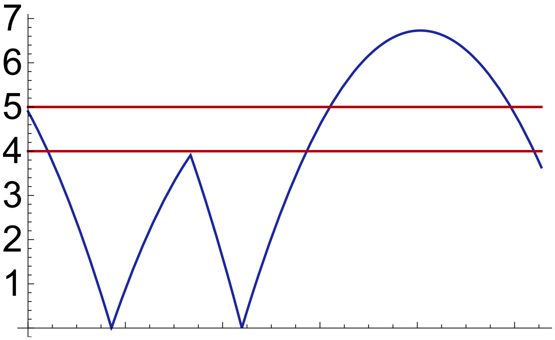
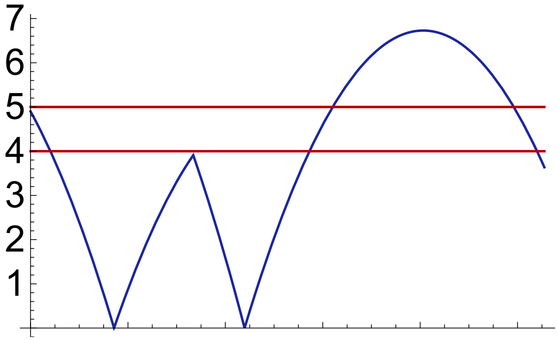


# Quantum Discovers Design-by-Invariant

## Design-by-Invariant

$2gx = 2g{\color{red}H} - v^2 \land x \geq 0$



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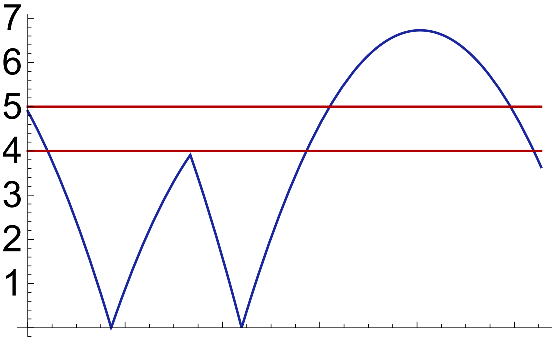
$2gx = 2g\textcolor{red}{5} - v^2 \land x \geq 0$



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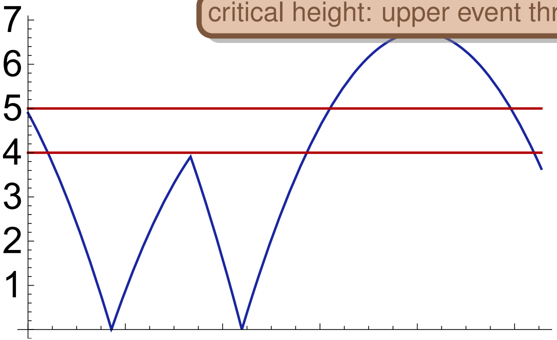
$2gx > 2g5 - v^2 \wedge x \geq 0$

critical height: upper event threshold



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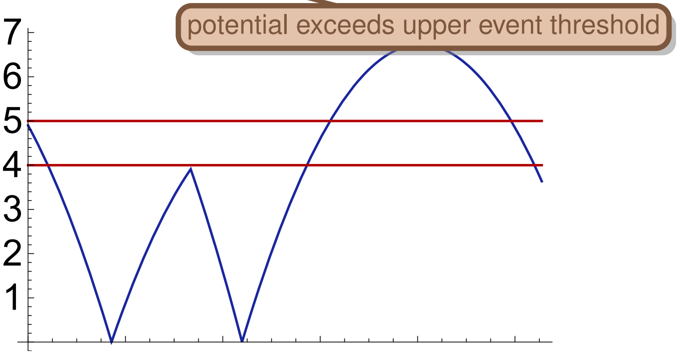
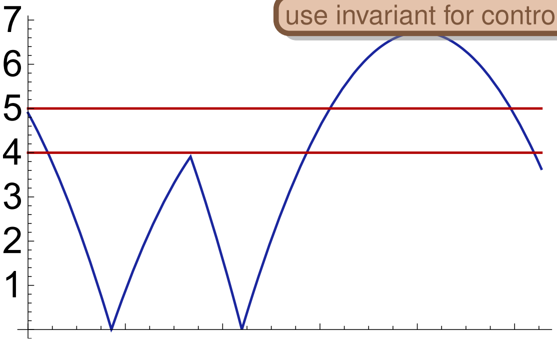


# Quantum Discovers Design-by-Invariant

## Design-by-Invariant

$2gx > 2g5 - v^2 \land x \geq 0$

use invariant for control



# Quantum the Time-triggered Ping-Pong Ball

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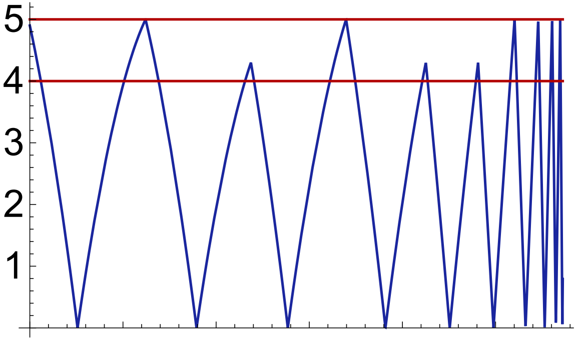
**Proof?** Ask René Descartes



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Just for simplicity

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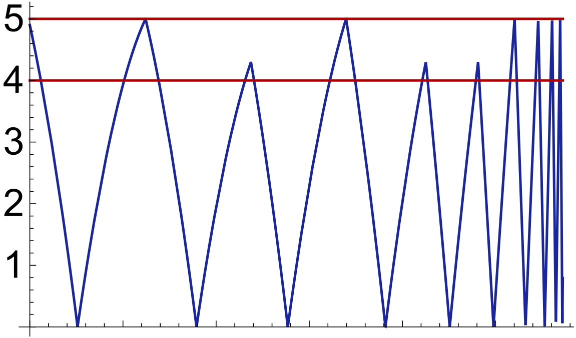

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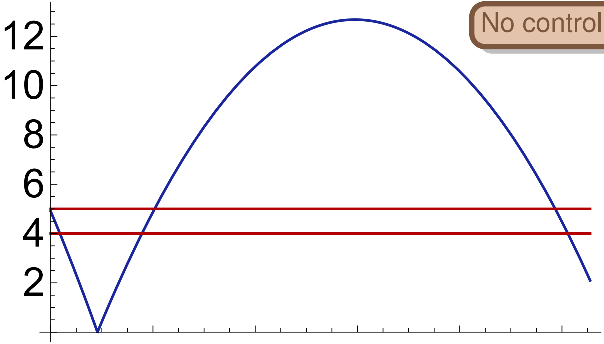
**Proof?** Ask René Descartes who says no!

No control when ball is on the ground



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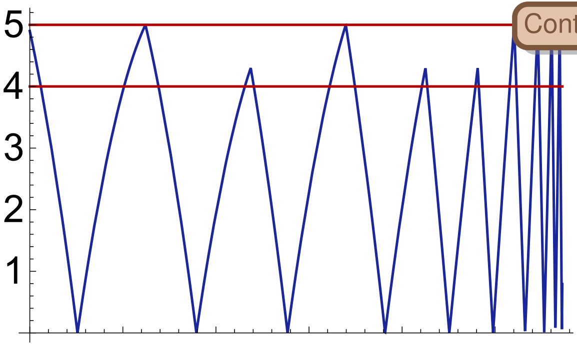
**Proof?** Ask René Descartes

Control despite ground



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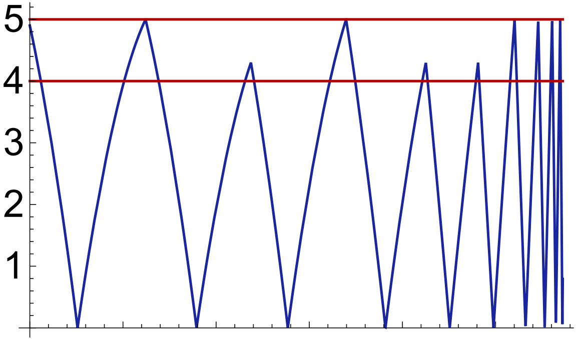

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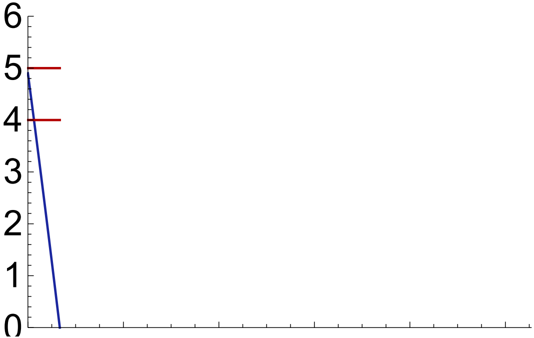
**Proof?** Ask René Descartes who says yes but should have said no!



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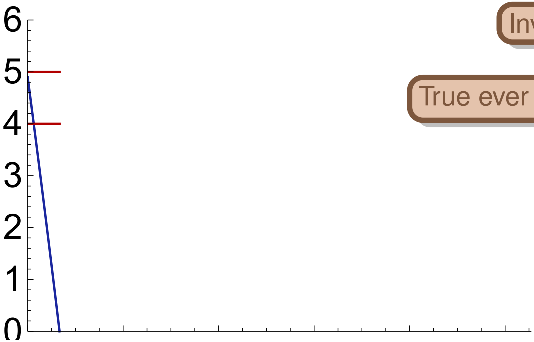
Invariants are **invariants**!

True ever $\rightsquigarrow$ true always $\rightsquigarrow$ eager action



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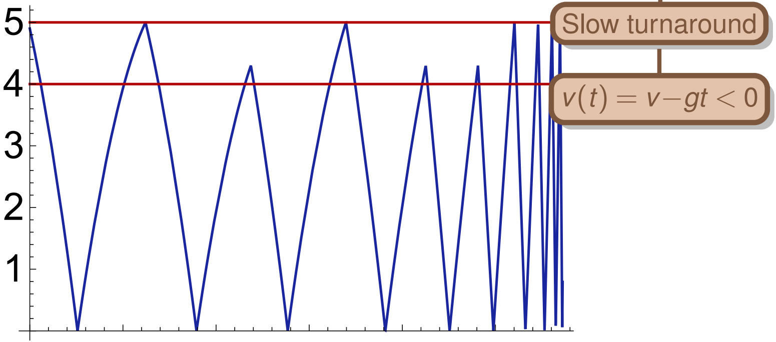
$$t := 0; \{x' = v, v' = -g, t' = 1 \,\&\, x \ge 0 \land t \le 1\}\big)^*\Big](0 \le x \le 5)$$

**Proof?** Ask René Descartes

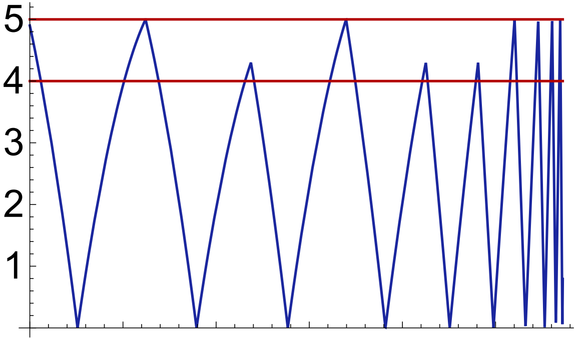


# Quantum the Time-triggered Ping-Pong Ball

## Conjecture (Quantum can play ping-pong safely)

$$0 \le x \land x \le 5 \land v \le 0 \land g = 1 \land 1 = c \land 1 = f \rightarrow$$
$$\Big[\Big(\text{if}(x{=}0)\, v := -cv; \text{if}((x{>}5+\tfrac{g}{2}-v \lor 2gx{>}2g5-v^2 \land v{<}g) \land v{\ge}0)\, v := -fv;$$
$$t := 0; \{x' = v, v' = -g, t' = 1 \,\&\, x \ge 0 \land t{\le}1\}\Big)^*\Big](0{\le}x{\le}5)$$

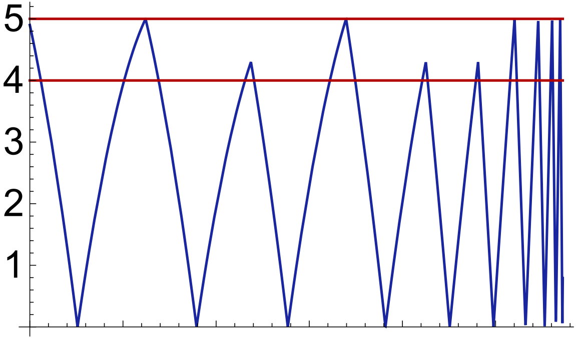
Proof? Ask René Descartes who says yes



**Proposition (Quantum can play ping-pong safely in real-time)**

$0 \le x \land x \le 5 \land v \le 0 \land g{=}1{>}0 \land 1{=}c{\ge}0 \land 1{=}f{\ge}0 \to$
$\big[\big(\text{if}(x{=}0)\, v := -cv; \text{if}((x{>}5 + \frac{g}{2} - v \lor 2gx{>}2g5 - v^2 \land v{<}g) \land v{\ge}0)\, v := -fv;$
$t := 0; \{x' = v, v' = -g, t' = 1 \,\&\, x \ge 0 \land t \le 1\}\big)^*\big](0{\le}x{\le}5)$

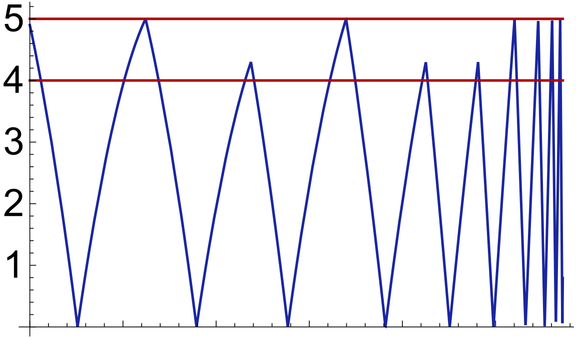
**Proof**



**Proposition (Quantum can play ping-pong safely in real-time)**

$$0 \le x \land x \le 5 \land v \le 0 \land g{=}1{>}0 \land 1{=}c{\ge}0 \land 1{=}f{\ge}0 \rightarrow$$
$$\big[\big(\text{if}(x{=}0)\, v := -cv; \text{if}((x{>}5+\tfrac{g}{2}{-}v \lor 2gx{>}2g5{-}v^2{\land}v{<}g){\land}v{\ge}0)\, v := -fv;$$
$$t := 0; \{x' = v, v' = -g, t' = 1 \,\&\, x \ge 0 \land t \le 1\}\big)^*\big](0{\le}x{\le}5)$$

**Proof** **@invariant**$(2gx = 2gH - v^2 \land x \ge 0 \land x{\le}5)$

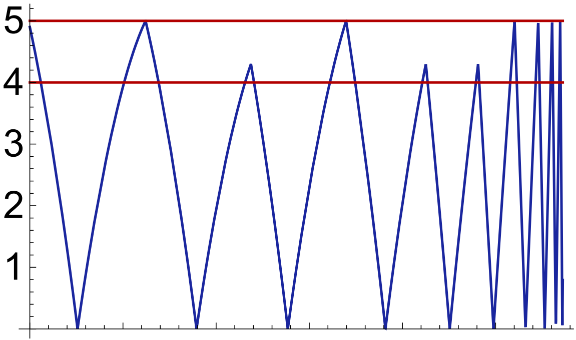


# Quantum's Time-triggered Ping-Pong Proof Invariants

**Proposition** (Quantum can play ping-pong safely in real-time)

$$2gx = 2gH - v^2 \land 0 \le x \land x \le 5 \land v \le 0 \land g{=}1{>}0 \land 1{=}c{\ge}0 \land 1{=}f{\ge}0 \rightarrow$$
$$\Big[\Big(\text{if}(x{=}0)\, v := -cv; \text{if}((x{>}5 + \tfrac{g}{2} - v \lor 2gx{>}2g5 - v^2 \land v{<}g) \land v{\ge}0)\, v := -fv;$$
$$t := 0; \{x' = v, v' = -g, t' = 1 \,\&\, x \ge 0 \land t \le 1\}\Big)^*\Big](0{\le}x{\le}5)$$

**Proof** $\textbf{@invariant}(2gx = 2gH - v^2 \land x \ge 0 \land x{\le}5)$



# Outline

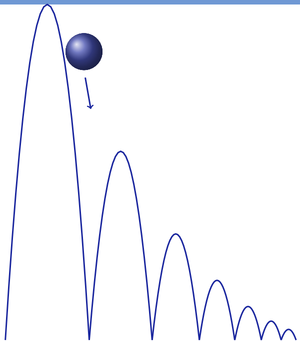
# Summary: Time-triggered Control

1. Common paradigm for designing real controllers
2. Periodical or pseudo-periodical control (jitter)
3. Expects delays, expects inertia
4. Implementation: discrete-time sensing
5. Predict events, not just: if($eventnow(x)$) ...
6. Safe controllers know their own reaction delays
7. Burden of event detection brought to attention of CPS programmer
8. Time-triggered controls are implementable and more robust, but make design and verification more challenging!
9. Use knowledge gained from verified event-triggered model as a basis for designing a time-triggered controller

# Outline

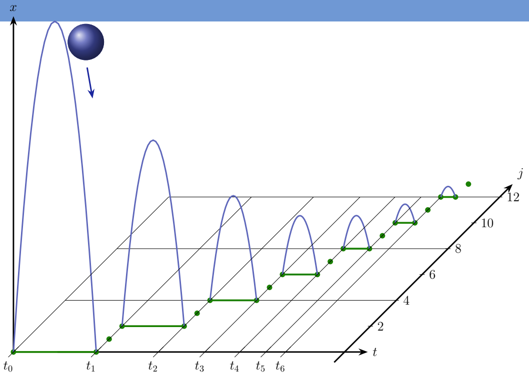
4. Appendix
   - Zeno's Quantum Turtles

# How Quantum Met Achilles and His Tortoise



**Example (Quantum the Bouncing Ball)**

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
$$\text{if}(x = 0)\, v := -cv\big)^*$$




## Example (Quantum the Bouncing Ball)

$$\big(\{x' = v, v' = -g \,\&\, x \geq 0\};$$
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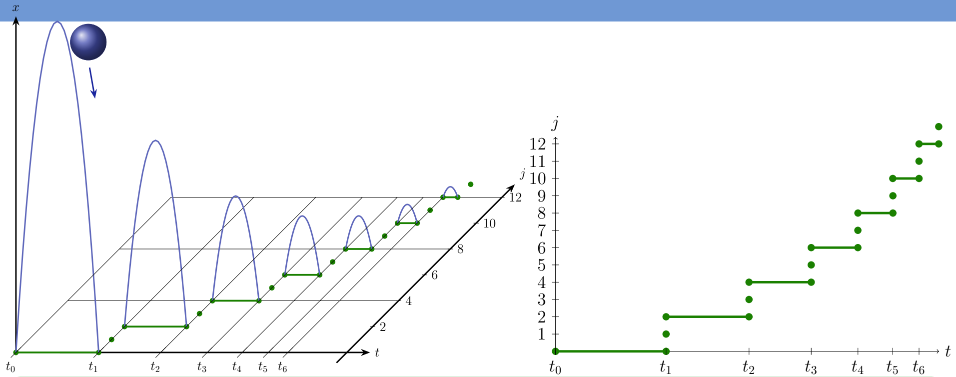
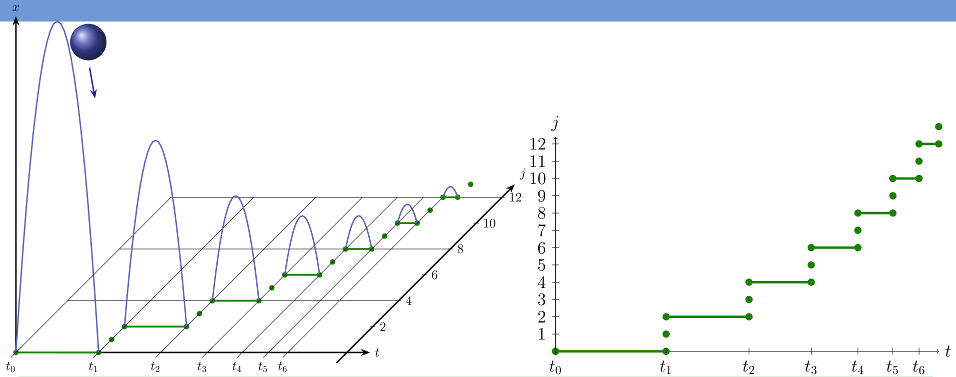



**Example (Quantum the Bouncing Ball)**

$$\left(\{x' = v, v' = -g \,\&\, x \geq 0\};\right.$$
$$\left.\text{if}(x = 0)\, v := -cv\right)^*$$

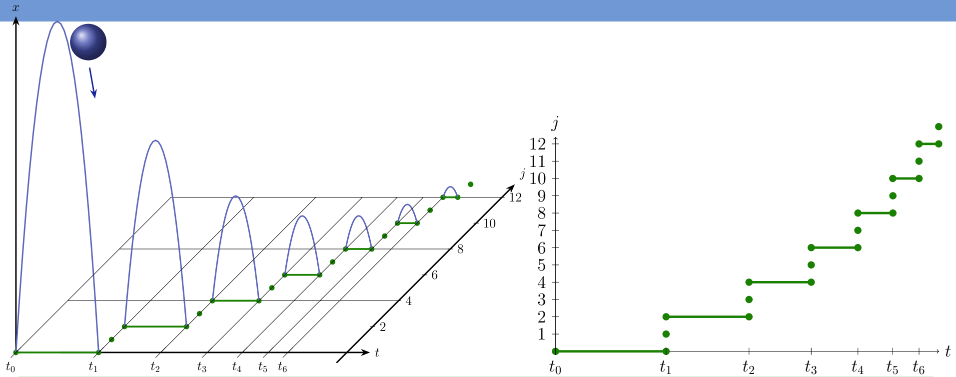
# How Quantum Met Achilles and His Tortoise




## Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

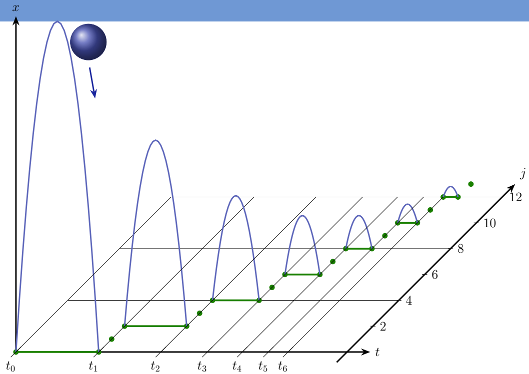
# How Quantum Met Achilles and His Tortoise

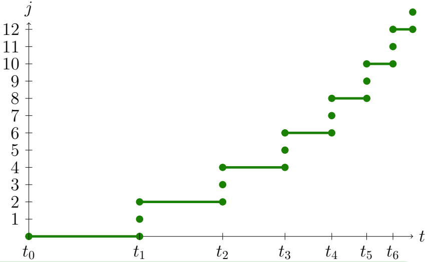



**Example (Quantum the Bouncing Ball experiences time)**

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i}$$

# How Quantum Met Achilles and His Tortoise

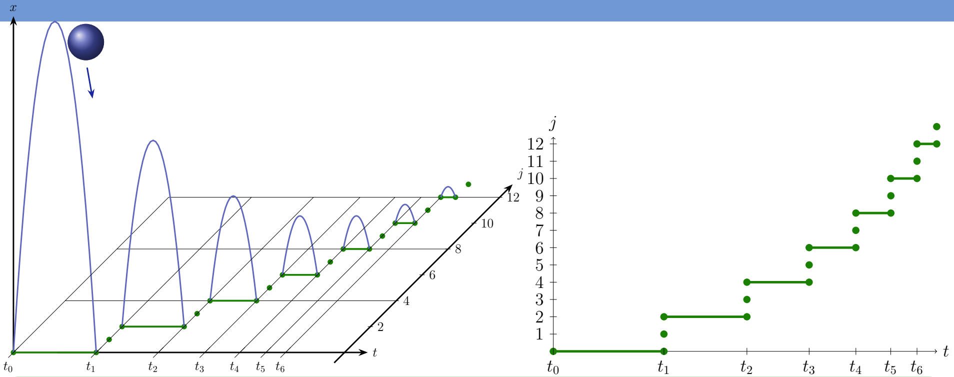



## Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}}$$

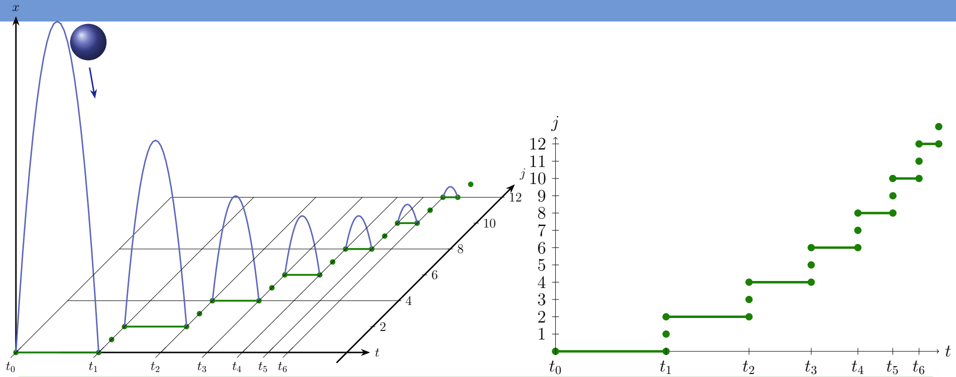
# How Quantum Met Achilles and His Tortoise




## Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}} = 2$$

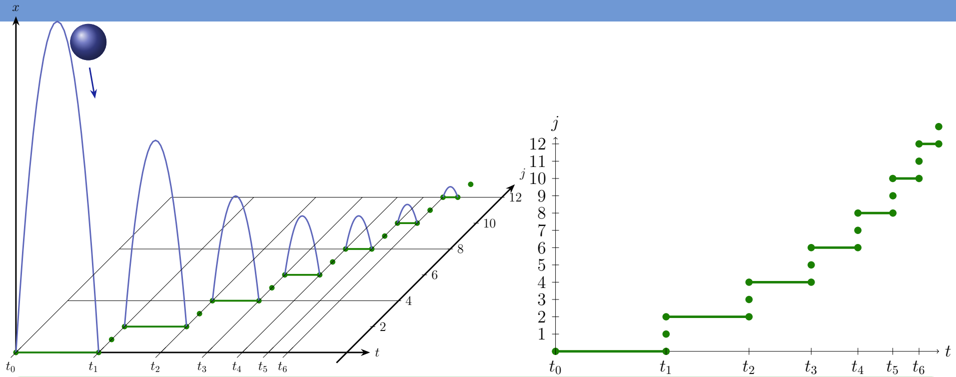
# How Quantum Met Achilles and His Tortoise




**Example (Quantum the Bouncing Ball experiences time)**

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=\sum_{i=0}^{\infty}\frac{1}{2^i}=\frac{1}{1-\frac{1}{2}}=2<\infty$$

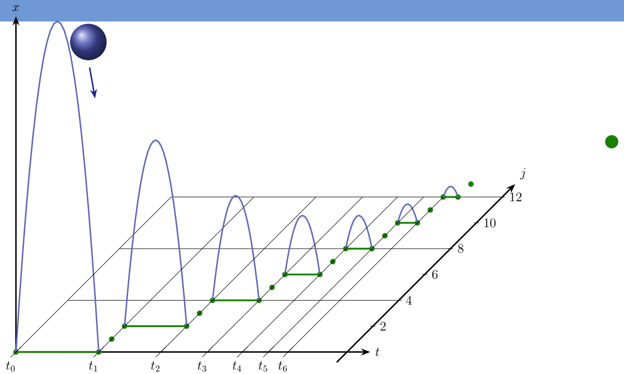
# How Quantum Met Achilles and His Tortoise




## Example (Quantum the Bouncing Ball experiences time)

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}} = 2 < \infty$$

# How Quantum Met Achilles and His Tortoise

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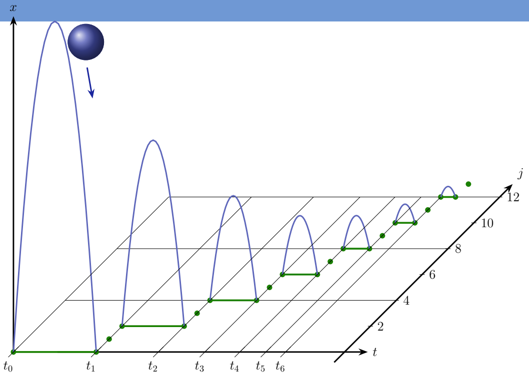
# How Quantum Met Achilles and His Tortoise

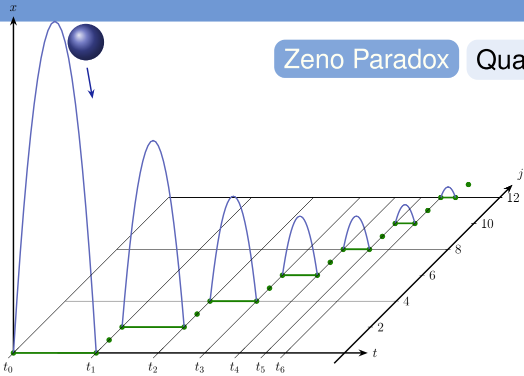


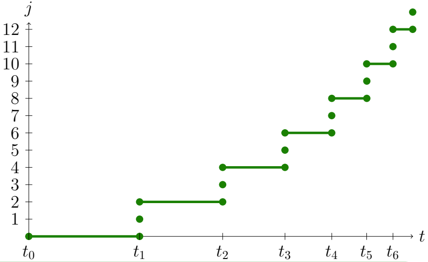


**Example (Quantum the Bouncing Ball experiences time)**

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=\sum_{i=0}^{\infty}\frac{1}{2^i}=\frac{1}{1-\frac{1}{2}}=2<\infty$$

**Zeno Paradox** Quantum's model causes a time freeze




**Example (Quantum the Bouncing Ball experiences time)**

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - \frac{1}{2}} = 2 < \infty$$

André Platzer.
*Logical Foundations of Cyber-Physical Systems*.
Springer, Switzerland, 2018.
URL: http://www.springer.com/978-3-319-63587-3, doi:10.1007/978-3-319-63588-0.

André Platzer.
*Logical Analysis of Hybrid Systems: Proving Theorems for Complex Dynamics*.
Springer, Heidelberg, 2010.
doi:10.1007/978-3-642-14509-4.

Sarah M. Loos and André Platzer.
Differential refinement logic.
In Martin Grohe, Eric Koskinen, and Natarajan Shankar, editors, *LICS*, pages 505–514, New York, 2016. ACM.
doi:10.1145/2933575.2934555.