15-859FF: Coping with Intractability **CMU, Fall 2019**
Lecture #14: Planar Graphs and Planar Separators October 21, 2019
Lecturer: Jason Li Scribe: Rudy Zhou

1 Introduction

In this lecture, we will consider algorithms for graph problems on a restricted class of graphs - namely, planar graphs. We will recall the definition of planar graphs, state the key property of planar graphs that our algorithms will exploit (planar separators), apply planar separators to maximum independent set, and prove the Planar Separator Theorem.

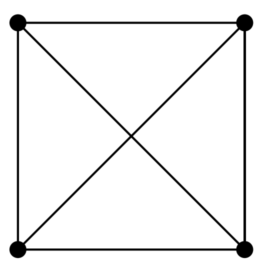
2 Planar Graphs and Planar Separators

Informally speaking, a planar graph G is one that we can draw on a piece of paper such that its edges do not cross.

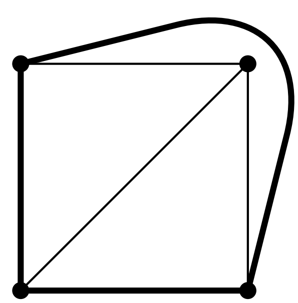
Definition 14.1 (Planar Graph)**.** An undirected graph G is planar if it admits a *planar drawing.* A planar drawing is a drawing of G in the plane such that the vertices of G are points in the plane, and the edges of G are curves connecting their respective endpoints (these curves do not have to be straight lines.) Further, we require that the edges do not cross, so they only intersect at possibly their endpoints.

On planar graphs, many of our intuitions about graphs that come from drawing them on paper actually do hold. In particular, fix a planar drawing of the planar graph G. Any cycle of G divides the plane into two parts: the area outside the cycle, and the area inside the cycle. We call a chordless cycle of G a *face* of G. In addition, G has exactly one *outer face* that is unbounded on the outside. The faces of G define a partition of the plane.

Example 14.2. K_4 (the complete graph on four vertices) is planar. Note that the most obvious drawing of K_4 is not planar.



(a) Non-planar drawing of K_4



(b) Planar drawing of K_4. Outer face in bold.

One useful property of planar graphs that is not immediate is that every planar graph can be broken up into relatively balanced connected components by only removing a small portion of its vertices. The formal theorem statement is the following:

Theorem 14.3 (Planar Separator Theorem [LT77]). *Every planar graph G on n vertices has a subset $S \subset V(G)$ of $O(\sqrt{n})$ vertices such that every connected component of $G - S$ has at most $\frac{2}{3}n$ vertices.*

We call such a set $S \subset V(G)$ that satisfies the conditions of Theorem 14.3 a *planar separator*.

3 Application: Maximum Independent Set

Before proving the existence of planar separators, we give some motivation about why they are useful in algorithm design. Using planar separators, we give an algorithm for the maximum independent set problem that runs in time $2^{O(\sqrt{n}\log n)}$.

In the maximum independent set problem, we are given as input an undirected graph $G = (V, E)$, and the goal is to output the largest independent set in G. We consider the case where G is a planar graph. Note that even if G is planar, then this problem is still NP-hard. We will design an algorithm using planar separators to prove the following theorem.

Theorem 14.4. *There exists an algorithm that solves the maximum independent set problem on planar graphs in time $2^{O(\sqrt{n}\log n)}$.*

The high-level idea of the algorithm is to use planar separators[1] to divide the graph into pieces and then divide-and-conquer on the pieces.

Algorithm Let $G = (V, E)$ be in the input planar graph on n vertices. We first recursively define a binary tree whose nodes each store a subgraph of G (we distinguish *nodes* of the recursion tree with *vertices* of the input graph.)

Let the root of the tree, r, store the whole graph, $G_r = G$. For an arbitrary node x in the tree storing graph G_x with vertex set V_x, we recursively define its two children ℓ and r as follows:

Apply Theorem 14.3 to find a planar separator in G_x, say $S_x \subset V_x$ of size $O(\sqrt{n})$. Then every connected component of $G_x - S_x$ has at most $\frac{2}{3}|V_x|$ vertices. We can partition these connected components into two groups, such that both groups have at most $\frac{2}{3}|V_x|$ vertices. Define one of these groups as G_ℓ and the other as G_r (see Figure 14.1.) Note that $|V_\ell|, |V_r| \leq \frac{2}{3}|V_x|$. We repeat this procedure until the leaves of the recursion tree have graphs with at most $\sqrt{n}$ vertices each. Let T be the final tree. We summarize the properties of T in the following lemma:

Lemma 14.5. *The recursion tree T has the following properties:*

1. *T has depth $O(\log n)$ and $O(\sqrt{n}\log n)$ nodes.*
2. *For each node x, let $N(V_x) \subset V$ be the set of neighbors of V_x in G excluding V_x. Formally, $N(V_x) := \{v \in V \setminus V_x \mid vu \in E \text{ for some } u \in V_x\}$. Then $|N(V_x)| = O(\sqrt{n}\log n)$.*

Proof. The first property is immediate from the definition of T, so we prove $|N(V_x)| = O(\sqrt{n}\log n)$ for all nodes x. Let $v \in N(V_x)$. Note that v cannot be in V_x or any of x's descendants. Let a be the nearest ancestor of x in T such that $v \in V_a$. Then v is not in any descendant of a, so we conclude $v \in S_a$. This implies that every vertex in $N(V_x)$ is located in some separator of an ancestor of x. The node x has $O(\log n)$ ancestors and each ancestor has a separator of size $O(\sqrt{n})$. $\square$

[1] For now, we ignore the issue of making the Planar Separator Theorem algorithmic - just assume we can find such a separator in unit time.

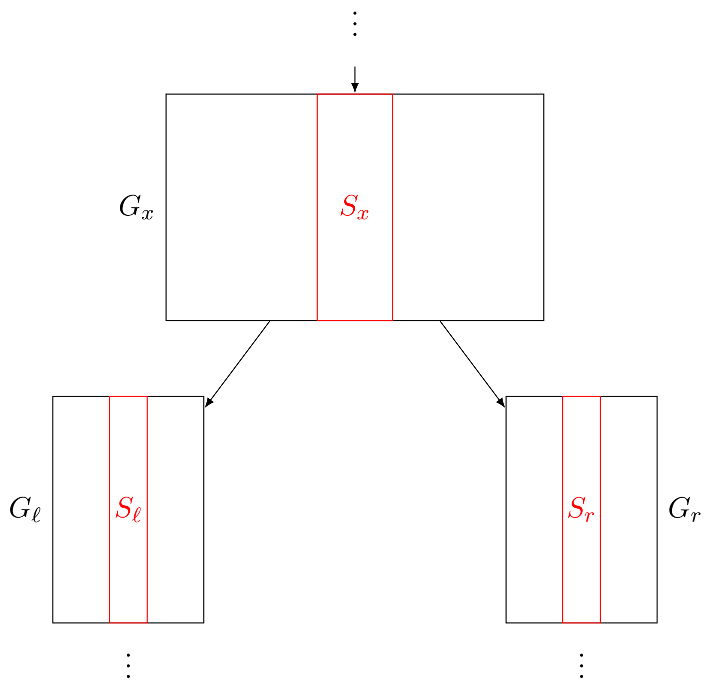


Figure 14.1: One node x with children ℓ and r of the recursion tree T

Now with our recursion tree in hand, we will compute the max independent set of G by a dynamic program. The subproblems are $P(x, X)$ for every node x of T and every subset $X \subset N(V_x)$ such that $P(x, X)$ is a maximum independent set $I \subset V_x \cup N(V_x)$ in $G[V_x \cup N(V_x)]$ such that $I \cap N(V_x) = X$. We define $P(x, X)$ recursively as follows:

The base case is when x is a leaf of the recursion tree, so $|V_x| = O(\sqrt{n})$. Thus we have $|V_x \cup N(V_x)| = O(\sqrt{n} \log n)$, so to compute $P(x, X)$, we can simply enumerate all possible subsets of $V_x \cup N(V_x)$ to find the desired independent set. This takes time $2^{O(\sqrt{n} \log n)}$ for each $X \subset N(V_x)$.

Otherwise, x has children, say ℓ and r. Then we define:

$$P(x, X) := P(\ell, (X \cup Y) \cap N(V_\ell)) \cup P(r, (X \cup Y) \cap N(V_r)),$$

where $Y \subset S_x$ is the subset of S_x that maximizes $|P(\ell, (X \cup Y) \cap N(V_\ell)) \cup P(r, (X \cup Y) \cap N(V_r))|$.

Conceptually, this recursion tries to glue together a solution I_ℓ on $G[V_\ell \cup N(V_\ell)]$ with a solution I_r on $G[V_r \cup N(V_r)]$ to get a solution I on $G[V_x \cup N(V_x)]$ subject to the constraint that $I \cap N(V_x) = X$. We will sketch the proof of this recusion in the following section.

In summary, our overall algorithm will be to recursively compute T, recursively compute $P(r, \emptyset)$, and output the resulting set.

Proof Sketch for Theorem 14.4 The runtime of the algorithm is dominated by computing $P(x, X)$ for all nodes x and all $X \subset N(V_x)$. It is easy to check that there are $2^{O(\sqrt{n} \log n)}$ subproblems and each can be computed in time $2^{O(\sqrt{n} \log n)}$.

To prove the correctness, it suffices to show that $P(x, X)$ correctly computes a maximum independent set $I \subset V(G_x) \cup N(G_x)$ of the graph $G[V(G_x) \cup N(V_x)]$ subject to the constraint that $I \cap N(V_x) = X$. The base case is trivial.

To prove the recursive case for $P(x, X)$, where x has children ℓ and r, note that we construct an independent set in $G[V_x \cup N(V_x)]$ using some vertices from V_ℓ, V_r, S_x, and $N(V_x)$. Let I^* be a maximum independent set in $G[V_x \cup N(V_x)]$ with $I^* \cap N(V_x) = X$. Further, consider the correct guess of $Y = I^* \cap S_x$.

The vertices we choose from S_x and $N(V_x)$ are fixed to be Y and X, respectively, so it remains to choose vertices from V_ℓ and V_r. While we cannot choose an independent set in G_ℓ directly, we can choose an independent set in $G[V_\ell \cup N(V_\ell)]$ that is consistent with the requirements $I^* \cap S_x = Y$ and $I^* \cap N(V_x) = X$. By the inductive hypothesis, the term $P(\ell, (X \cup Y) \cap N(V_\ell))$ chooses exactly such a set (and analogously for r.)

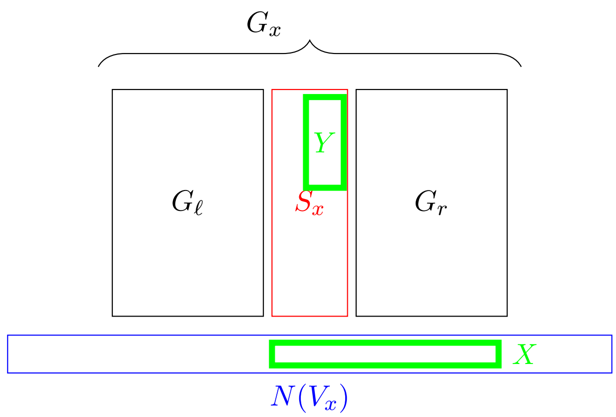


Figure 14.2: Illustration for computing $P(x, X)$, where x has children ℓ and r

4 Proof of Planar Separator Theorem

In this section we prove Theorem 14.3. We present a combinatorial proof due to Alon, Seymour, and Thomas [AST94], and also sketch how to make the proof algorithmic.

Preliminaries Throughout this section, let $G = (V, E)$ be a planar graph. Fix a planar drawing of G. We assume without loss of generality that G has no parallel edges or self loops, because such edges do not effect the planar separator. Further, we assume that G is *triangulated*, so every face of G (including the outer face) is a triangle. This can be achieved by adding all chord edges to all faces of G. Observe that a planar separator for the triangulation of G is a planar separator for G, because G is a subgraph of its triangulation.

Notations For a simple cycle C in G, we define the sets $In(C), Out(C) \subset V$ such that $In(C)$ is the set of all vertices of G that are strictly inside the cycle C, and $Out(C)$ is the set of all vertices of G that are strictly outside the cycle C. Here is one example of an intuitive property that actually does hold because of planarity: removing C from G disconnects $In(C)$ from $Out(C)$, because no edges can cross C, so any path between $In(C)$ and $Out(C)$ must include a vertex on C.

Using these notations, we define the disk of C by $D_C := G[V(C) \cup In(C)]$ to be the subgraph of G corresponding to the cycle C and all vertices inside C. We will omit the subscript C when it clear from context.

Finally, let $k := \sqrt{n+1}$ for convenience.

Proof Overview It turns out, the planar separator we seek is actually a cycle C of G.[2] Because deleting C from G disconnects $In(C)$ from $Out(C)$ it suffices to find a C satisfying:

1. $|V(C)| \leq 4k$
2. $|Out(C)| \leq \frac{2}{3}n$
3. $|In(C)| \leq \frac{2}{3}n$

It is not immediately clear that such a cycle exists or how to find one, but as a start, note that the outer face of G already satisfies properties 1 and 2, because if C_0 is the outer face of G, then $|V(C_0)| = 3$ because G is triangulated and $|Out(C)| = 0$, because by definition, there are no vertices outside the outer face.

Unfortunately, $|In(C_0)|$ is not at most $\frac{2}{3}n$ in general. How can we get around this? The goal in the proof of Theorem 14.3 is to show that among all cycles that satisfy 1 and 2, there exists one that also satisfies 3. We cannot impose 3 directly (that is what we are trying to prove!), so instead we replace 3 with a proxy that must exist. In particular, we impose:

4. C minimizes $\Delta(C) := |In(C)| - |Out(C)|$ over all cycles satisfying properties 1 and 2

Observe that there must exist such a cycle C that satisfies 1, 2, and 4! We will show that this C is indeed the planar separator we seek by using the property that C minimizes $\Delta(\cdot)$.

The Proof Our proof of Theorem 14.3 will make use of Menger's theorem for minimum vertex cuts. We state it here:

Theorem 14.6 (Menger's Theorem [Men27]). *Let G be a graph. For any subsets $A, B \subset V(G)$, the size of the minimum vertex cut between A and B is equal to the number of vertex-disjoint paths from A to B.*

Proof of Theorem 14.3. Let C be a cycle in G satisfying 1, 2, and 4. As argued in the Proof Overview, such a C exists (take the outer face of G), and it suffices to show that $|In(C)| \leq \frac{2}{3}n$. This will be the goal for the remainder of the proof.

We assume for contradiction that $|In(C)| > \frac{2}{3}n$. This assumption gives us the following two claims.

Claim 14.7. *For all $u, v \in C$, we have $d_D(u, v) = d_C(u, v)$*

Proof. The direction $d_D(u, v) \leq d_C(u, v)$ is immediate because C is a subgraph of D. Thus it suffices to show that $d_D(u, v) \geq d_C(u, v)$ for all $u, v \in C$. We assume for contradiction that there exist $u, v \in C$ such that $d_D(u, v) < d_C(u, v)$.

Without loss of generality, we may assume that the path in D from u to v that achieves distance $d_D(u, v)$ has all of its internal nodes in $In(C)$. If this is not the case, then the path consists of sections that go along the cycle and sections that only use internal nodes. One of these internal node sections has endpoints, say $u'v' \in C$, with $d_D(u', v') < d_C(u', v')$. Then consider u', v' instead of u, v.

Let this path be σ. Observe that σ divides C into two cycles, say C^+ and C^-, where we may assume $|In(C^+)| \geq |In(C^-)|$ (see Figure 14.3.) We claim that C^+ also satisfies 1 and 2, and $\Delta(C^+) < \Delta(C)$. Note that this is a contradiction, because by definition $\Delta(C) \leq \Delta(C')$ for all cycles C' satisfying 1 and 2, so it remains to show that C^+ has the desired properties.

[2] This is assuming that G is triangulated, of course. If we start off with a graph and triangulate it, the cycle may contain triangulated edges, so it may not be a cycle in the original graph once we remove these additional edges.

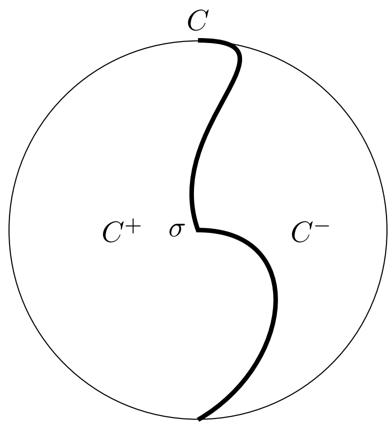


Figure 14.3: Cycle C split into C^+ and C^- by the path σ (bolded.)

1. We compute:

$$\begin{aligned} |V(C^+)| &= |V(C^+ \setminus \sigma)| + d_D(u,v) \\ &< |V(C^+ \setminus \sigma)| + d_C(u,v) \\ &\leq |V(C^+ \setminus \sigma)| + |V(C^-)| \\ &= |V(C)| \leq 4k \end{aligned}$$

2. We recall that by assumption, $|In(C)| > \frac{2}{3}n$ and $|In(C^+)| \geq |In(C^-)|$. Also, because σ is a subgraph of C^+, we have $|V(\sigma)| \leq |V(C^+)|$. Combining these bounds, we compute:

$$\begin{aligned} |In(C)| &= |In(C^+)| + |In(C^-)| + |V(\sigma)| - 2 \\ \frac{2}{3}n &< 2|In(C^+)| + |V(C^+)| - 2, \end{aligned}$$

which implies:

$$|In(C^+)| > \frac{1}{3}n - \frac{1}{2}|V(C^+)| + 1$$

Using this result, we can bound the size of $Out(C^+)$ by the following relation:

$$\begin{aligned} |Out(C^+)| &= n - |V(C^+)| - |In(C^+)| \\ &< \frac{2}{3}n + \frac{1}{2}|V(C^+)| - 1 \leq \frac{2}{3}n, \end{aligned}$$

where in the final inequality we use the fact that C^+ has at least three vertices because it is a cycle.

4. Because C^+ is inside C, we have $|Out(C^+)| \geq |Out(C)|$ and $|In(C^+)| < |In(C)|$. These two inequalities give:

$$\Delta(C^+) = |In(C^+)| - |Out(C^+)| < |In(C)| - |Out(C)| = \Delta(C).$$

□

Claim 14.8. *C has exactly $4k$ vertices.*

Proof. We assume for contradiction that $|V(C)| < 4k$. Then pick an arbitrary edge along the cycle C, say uv. Because G is triangulated, the edge uv is incident on exactly two faces in G: one (weakly) inside C and one outside C. Let w be the unique vertex weakly inside C such that uvw is a face of G. There are two cases to consider - either $w \in V(C)$, so w is on the cycle C, or $w \in In(C)$, so w is strictly inside C.

In the former case, uw is a chord for the cycle C, so $d_D(u, w) = 1$ because $uw \in E(D)$, but $d_C(u, w) \geq 2$ because uw is a chord for the cycle C. This contradicts Claim 14.7.

It remains to consider the case $w \in In(C)$. The cycle C is $u \to v \to \cdots \to u$. Then we define a new cycle C' by modifying C such that C' takes the edges uw and wv instead of edge uv as in C. Then C' is $u \to w \to v \to \cdots \to u$. We claim that C' also satisfies 1 and 2, and $\Delta(C') < \Delta(C)$. This is a contradiction, so it remains to check that C' has the desired properties:

1. We have $|V(C)| < 4k$ and $|V(C')| = |V(C)| + 1$, so $|V(C')| \leq 4k$, as required.

2. By definition of C', uvw is a face of G, so there is no vertex inside uvw. Thus $|Out(C')| = |In(uvw)| + |Out(C)| = |Out(C)| \leq \frac{2}{3}n$

4. We have $|Out(C')| = |Out(C)|$, but $|In(C')| < |In(C)|$, because w in inside C but not C'. This immediately implies $\Delta(C') < \Delta(C)$.

□

Now using the above claim, we can split the cycle C into four disjoint sub-paths of exactly k vertices each. Let these paths have vertex sets L, R, T, B (for left, right, top, and bottom, respectively.) See Figure 14.4. Consider the minimum vertex cut between L and R in D. Because G is planar, this cut must break each path that goes from L to R inside D. To do so, the cut must contain a set of vertices forming a path that connects T and B. This is another example of an intuitive property that holds because of planarity, because every path from L to R in D must cross this path from T to B. Let τ be such a path from a vertex in $t \in T$ to a vertex in $b \in B$ such that all vertices in τ are in the cut.

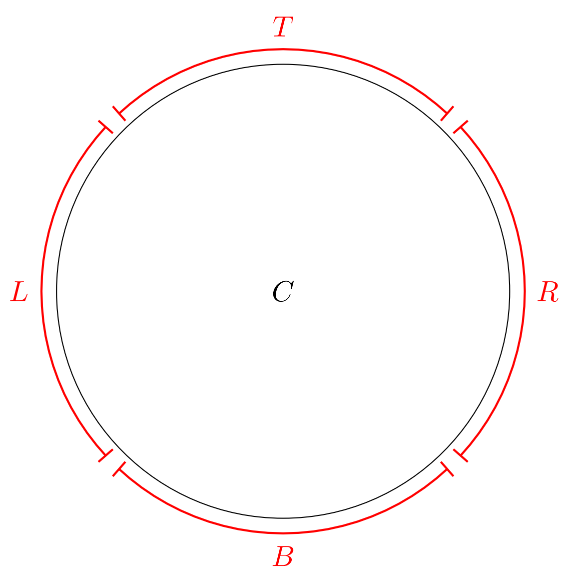


Figure 14.4: Cycle C split into sub-paths L, R, T, B - each with exactly k vertices.

By Claim 14.7, $|V(\tau)| - 1 \geq d_D(t, b) = d_C(t, b) \geq k$, because in order to get from t to b in C, we must go through the entirety of either L or R. Note that $V(\tau)$ is a subset of the minimum vetex cut between L and R in D, so by applying Theorem 14.6, we can lower bound the number of vertex-disjoint paths from L to R in D by k.

Each of these k paths has length at least k, again by applying Claim 14.7. However, this implies that $|V(D)| \geq k^2 = n + 1$. This is a contradiction since there are only n vertices in the graph, so we conclude that $|In(C)| \leq \frac{2}{3}n$, so C is the desired planar separator. □

Making the Proof Algorithmic We sketch a way to turn the above proof into an algorithm to find such a planar separator. Start with C as the outer face of G. If C satisfies $|In(C)| \leq \frac{2}{3}n$, then we have found our separator! Otherwise, $|In(C)| > \frac{2}{3}n$, so as in Claim 14.7, we can find two smaller cycles C^+, C^- in C such that $\Delta(C^+) < \Delta(C)$ and C^+ still satisfies 1 and 2. Then we update $C \leftarrow C^+$ and repeat. Each iteration decreases $\Delta(C)$ by at least 1, and $\Delta(C) \leq n$, so this algorithm has $O(n)$ iterations.

References

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