HOMEWORK 2
**Due: Thursday, September 29**

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**Ground rules:** *same as for Homework 1.*

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**1. Unemployment.** Consider the *assignment problem* studied in class; i.e., Maximum-Weight Perfect Matching in a bipartite graph $G = (U, V, E)$.

(a) Suppose now that there are more people than jobs; i.e., $|U| > |V|$. We still want every job done, but some people will not be assigned any job. Formulate the appropriate integer program and LP relaxation.

(b) Show that the integrality theorem from class still holds: if the LP relaxation is feasible, then every extreme point is integral.

**2. George & Leslie's Theorem.** Recall the LP relaxation for Minimum Vertex-Cover:

$$\begin{aligned} \min \quad & \sum_{v \in V} c_v x_v \\ \text{s.t.} \quad & 0 \le x_v \le 1 \quad \text{for all } v \in V, \\ & x_u + x_v \ge 1 \quad \text{for all } (u, v) \in E. \end{aligned}$$

(a) Let $\widetilde{x}$ be any feasible solution for the LP. Define another solution $x^+$ by

$$x_v^+ = \begin{cases} \widetilde{x}_v + \epsilon & \text{if } \frac{1}{2} < \widetilde{x}_v < 1, \\ \widetilde{x}_v - \epsilon & \text{if } 0 < \widetilde{x}_v < \frac{1}{2}, \\ \widetilde{x}_v & \text{if } \widetilde{x}_v \in \{0, \frac{1}{2}, 1\}. \end{cases}$$

Similarly define the solution $x^-$, replacing $\epsilon$ with $-\epsilon$. Prove that one can find $\epsilon > 0$ such that both $x^+$ and $x^-$ are feasible for the LP. (Hint: there are at least four cases.)

(b) Show that every extreme point $x^*$ of the LP is *half-integral*, i.e. $x_v^* \in \{0, \frac{1}{2}, 1\}$ for all $v \in V$.

**3. Reductio ad solutionem de feasibility.** Consider the computational problem of solving a general LP $\min\{c^T x \mid Ax \ge b\}$; we'll call it SOLVE-LP. It takes as input the $m \times n$ matrix $A$, the vectors $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. The desired output is:

- `Infeasible` if the LP is infeasible,
- `Unbounded` if the optimal value is $-\infty$,
- or a vector $x \in \mathbb{R}^n$ which is an optimal feasible solution to the LP.

Give a reduction from SOLVE-LP to the decision version of polyhedron feasibility defined in Hwk1(#1). As always, your reduction should run in time polynomial in the input length. (Hint: recall Hwk1(#4) and the discussion about binary search in class.)

**4. A Farkas Lemma.**

(a) Show the execution of our LP-feasibility-testing algorithm ("Simplex with the $b$-rule") on the following system:

$$\begin{aligned} -x_1 + 2x_2 + x_3 &\le 3 \\ 3x_1 - 2x_2 + x_3 &\le -17 \\ -x_1 - 6x_2 - 23x_3 &\le 16 \\ x &\ge 0 \end{aligned}$$

This system is infeasible, so execution should end with an "offending equation" (i.e., a basic variable equated to a negative constant plus a nonpositive linear combination of nonbasic variables).

(b) Rearrange the offending equation so that the slack variables are on the LHS (with nonnegative coefficients), a negative constant is on the RHS, and the original variables are on the RHS (with nonpositive coefficients). Write out all six coefficients explicitly (even if some are 0 or 1).

(c) This offending equation is a unique linear combination of the equations in the initial tableau. Which linear combination? Why must it be unique?

(d) Take the same linear combination of the original *inequalities*. Why is the resulting inequality obviously incompatible with the constraint $x \ge 0$?

(e) Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Show that exactly one of the following is true:

- $\exists x \ge 0$ s.t. $Ax \le b$
- $\exists y \ge 0$ s.t. $y^\top A \ge 0$, $y^\top b < 0$.

**5. Max-Sat.** A *Sat instance* over Boolean variables $u_1, \dots, u_n$ consists of a list of $m$ *clauses*, each of which is a *disjunction* (OR) of *literals* (a variable $u_i$ or its negation $\overline{u}_i$). Let Opt denote the maximal number of clauses that can be satisfied by a Boolean assignment to the variables. The *Sat problem* is to determine whether or not Opt $= m$. The *Max-Sat problem* is to find an assignment satisfying Opt clauses. We remark that both tasks are NP-hard.

(a) Formulate an integer program (IP) capturing Max-Sat. There should be a 0-1 IP-variable for each Sat-variable, as well as a 0-1 IP-variable representing the "truth value" of each clause.

(b) Suppose your relax your IP to an LP. Find an instance of Max-Sat with 4 clauses for which Opt $= 3$ but LPOpt $= 4$.

(c) A *Horn-Sat instance* is a special kind of Sat instance in which each clause has at most one positive (i.e., unnegated) literal. The Sat problem restricted to Horn-Sat instances is in P (you might like to convince yourself of that), though the Max-Sat problem remains NP-hard. Show that your LP relaxation solves the Sat problem for Horn-Sat instances, in the sense that LPOpt $= m$ if and only if Opt $= m$.