Linear Programming and Semindefinite Programming **CMU 15-859E, Fall 2011**

HOMEWORK 1
Out: Tuesday, September 15
Due: Thursday, September 22

Groundrules

- Homeworks will sometimes consist of *exercises*, easier problems designed to give you practice, and *problems*, that may be harder, trickier, and/or somewhat open-ended. You should do the exercises by yourself, but you can work in groups with at most three people on the harder problems if you want. One exception: no fair working with someone who has already figured out (or already knows) the answer. If you work with others, then write down who you are working with.
- If you've seen a problem before (sometimes we'll give problems that are "famous"), then say that in your solution (it won't affect your score, we just want to know). Also, if you use any sources other than the textbook, write that down too (it's fine to look up a complicated sum or inequality or whatever, but don't look up an entire solution).

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0. Exercises. These will be posted on the blog by later tonight. You shouldn't turn in solutions to these, but make sure you do solve them.

1. To Search or to Decide: 'tis the Same. Consider the following two problems. For both the problems, the input is the specification of m halfspaces $\{a_i^T x \leq b_i\}_{i=1\ldots m}$, which defines a polyhedron $P = \cap_{i=1}^m \{x \mid a_i^T x \leq b_i\} \subseteq \mathbb{R}^n$.

(a) *(Polyhedron Feasibility: Decision Version)* The output is `Feasible` if P is non-empty, and `Infeasible` otherwise.

(b) *(Polyhedron Feasibility: Search Version)* The output is a point $x \in P$ if P is non-empty, and `Infeasible` otherwise.

Show the two problems are polynomial-time equivalent: i.e., given an algorithm $\mathcal{A}$ for one, show how to implement an algorithm for the other. Your algorithm should run in polynomial time, and make only $poly(m, n)$ calls to $\mathcal{A}$, each time calling $\mathcal{A}$ with only $O(m)$ constraints.

2. Some Tasks from Learning Theory. For the following problem you are given n "data points": each is a pair (X_i, y_i), where $X_i \in \mathbb{R}^d$ and $y_i \in \mathbb{R}$. (For intuition, you might like to think of the special case $d = 1$.)

(a) Suppose you wish to find the best-fitting hyperplane in "absolute error loss": this means you want to find the hyperplane $ax + b = 0$ (where $a \in \mathbb{R}^{1\times d}$ and $b \in \mathbb{R}$) which minimizes $\sum_{i=1}^n |aX_i + b - y_i|$. Formulate the task as a linear program.

(b) Suppose that each y_i is either $+1$ or -1. We say $ax + b = 0$ is a *weakly separating hyperplane* for the data points if $aX_i + b \geq 0$ whenever $y_i = +1$ and $aX_i + b \leq 0$ whenever $y_i = -1$. Formulate an LP which is feasible if and only if there is a weakly separating hyperplane.

(c) Continuing the above, say that $ax + b = 0$ is a *separating hyperplane* if $aX_i + b > 0$ whenever $y_i = +1$ and $aX_i + b < 0$ whenever $y_i = -1$. Show that linear programming can be used to decide if there is a separating hyperplane.

(d) Continuing the above, suppose there is no separating hyperplane. Show that the hyperplane which minimizes "hinge loss" can be found using linear programming. Here the hinge loss of $ax + b$ on a point $(X_i, +1)$ is defined to be 0 if $aX_i + b \geq 1$ and $1 - (aX_i + b)$ otherwise; the hinge loss on a point $(X_i - 1)$ is defined to be 0 if $aX_i + b \leq -1$ and $(aX_i + b) - (-1)$ otherwise.

(e) For all of the above problems, what if instead of hyperplanes we are looking for "quadratic surfaces"? This means equations of the form

$$\sum_{1 \leq k \leq \ell \leq d} c_{k\ell} x_k x_\ell + \sum_{1 \leq k \leq d} a_k x_k + b = 0.$$

Show that we can still solve all of the problems using polynomial-sized LPs.

3. Balls in Bins. You are given a region $K \subseteq \mathbb{R}^n$ defined by m linear inequalities: $K = \{x \mid Ax \leq b\}$. Your task is to find the largest sphere[1] S fitting inside K. Formulate a linear program which solves this problem, and explain why it is correct. (Be sure to explain what it means if your LP is infeasible or unbounded.)

4. Sizing Up the Solutions. For an integer k, define $\text{size}(k) = 1 + \lceil \log_2(|k| + 1) \rceil$; for a rational p/q (with p, q coprime, $q > 0$), define $\text{size}(p/q) = \text{size}(p) + \text{size}(q)$; for a matrix $R = (r_{ij})$ of rationals, define $\text{size}(M) = \sum_{i,j} \text{size}(r_{ij})$. Let $det(R)$ denote the determinant of R.

(a) If R is an $m \times m$ matrix, show that $size(det(R)) \leq poly(m, size(R))$.

Now consider the equational form LP $\min\{c^T x \mid Ax = b, x \geq 0\}$.

(b) If each of the numbers in A and b are rationals having size at most S, and if x^* is a basic feasible solution, give an upper bound on the size of each entry of x^*. In particular, show that each entry x_i^* is a rational number with size at most $K = \text{poly}(m \cdot S)$.

(c) If the LP has a finite optimum, and the numbers in c have size at most S, infer that the optimal value of the LP has size $O((K + S)m)$.

Hint: Google for 'Cramer's rule' and 'Hadamard Inequality'.

[1] Well, *a* largest sphere.