# Spectral Algorithm for Max-cut

Recall:

Given $G=(V,E)$, $\text{max-cut}(G) = \max_{S\subseteq V} \frac{|E(S,\bar{S})|}{m}$.

Notation: $E(S)$ = edges with both end points in $S$.

Goemans-Williamson: There's an algorithm that takes input a graph $G$ and outputs a cut $S\subseteq V$ such that $\text{cut}(S) = \frac{|E(S,\bar{S})|}{m}$ satisfies:

1) If $\text{max-cut}(G) \geq 1-\varepsilon$
then $\text{cut}(S) \geq 1 - O(\sqrt{\varepsilon})$.

2) $\text{cut}(S) \geq \alpha_{GW} \cdot \text{max-cut}(G)$

for $\alpha_{GW} \approx 0.878\ldots$

This algorithm is cool and uses semidefinite programming. We also discussed why linear programs are not useful for approximating max-cut better than $\frac{1}{2}$.

Last week, new technique: Spectral Methods

Today: Spectral methods qualitatively match the guarantees of Goemans-Williamson and provide a ~~0.61~~ factor approx.
$\sim 0.52 > 0.5$

Last time: Sparsest cut via smallest eigenvalue of Laplacian.

Today: Max-Cut via largest eigenvalue of Laplacian

We will focus only on regular graphs to keep exposition simple.

Then, $m = n \cdot d/2$

**Recall:** $\mathcal{L}_G \in \mathbb{R}^{n \times n}$ (Laplacian)

$\forall x \in \mathbb{R}^n$, $\quad x^T \mathcal{L}_G \cdot x = \sum_{\{i,j\} \in E} (x_i - x_j)^2$

If $x \in \{\pm 1\}^n$ s.t. $x_i = 1 \Leftrightarrow i \in S$, then, $x^T \mathcal{L}_G \cdot x = 4 |E(S, \bar{S})|$

(i.e. $x$ indicates $S$)

Thus, max-cut$(G)$

$$= \max_{x \in \{\pm 1\}^n} \frac{1}{2nd} x^T \mathcal{L}_G \cdot x.$$

$$= \max_{x \in \{\pm 1\}^n} \frac{1}{2d} \cdot \frac{x^T \mathcal{L}_G \cdot x}{\|x\|_2^2}$$

**Definition** $(\lambda_n)$.

$$\lambda_n = \max_{\substack{x : \|x\|_2 \neq 0 \\ x \in \mathbb{R}^n}} \frac{x^T \mathcal{L}_G \cdot x}{\|x\|_2^2}$$

the largest eigenvalue of $\mathcal{L}_G$.

Notice that up to a scaling by $1/2d$, $\lambda_n$ differs from max-cut only via dropping the "Booleanity" ($x \in \{\pm 1\}^n$) constraints. As $\lambda_2$ & sparsest cut, we'd like to relate $\lambda_n$ & max-cut.

In particular, would like $\lambda_n$ large $\Rightarrow$ max-cut is large.

Subtlety: Unlike $\lambda_2$, $\lambda_n$ is less "robust" in that it may fail to notice all but a very small part of the graph.

Obs$^n$: $\lambda_n \leq 2d$ $\forall$ $d$-regular $G$.

Proof: $\lambda_n \leq \max_i \|L_i\|_1 \leq 2d$

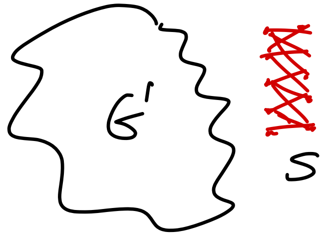
$i^{th}$ row of $L$

Prop: If $G$ is $d$-reg and has a connected component that is bipartite then

$$\lambda_n = 2d.$$

Proof: $S \subseteq V$.

Let $S = L, R$ — right vertex set; left vertex set




Choose $x_i = \begin{cases} 1 & \text{if } i \in L \\ -1 & \text{if } i \in R \\ 0 & \text{if } i \notin S \end{cases}$

$\lambda_n$ entirely governed by the bipartite component.

Then,

$$x^T \mathcal{L}_G \cdot x = \sum_{\{i,j\} \in E} (x_i - x_j)^2$$

$$= \sum_{\{i,j\} \in E(S)} (x_i - x_j)^2$$

$$= 4 \cdot \frac{|S| \cdot d}{2} = 2d|S|$$

Thus, $$\frac{x^T \mathcal{L}_G \cdot x}{\|x\|_2^2} = \frac{2|S| \cdot d}{|S|} = 2d.$$

$\square$

Since $\lambda_n$ can be influenced by a small "almost-bipartite" components, cannot hope to directly relate max-cut$(G)$ to $\lambda_n$.

But can still hope to prove a "stable" converse of above proposition.

"If $\lambda_n \geqslant 2d(1-\varepsilon)$, there should be a subset $S$ of vertices such that the induced graph on $S$ is almost bip, almost disc

almost bipartite $= \exists\, L, R : L \cup R = S$

s.t. $\dfrac{|E(L,R)|}{|E(S)|} \geqslant 1 - O(\sqrt{\varepsilon})$."

Def (bipartiteness gap)

$$\beta(G) = \min_{y \in \{\pm 1, 0\}^n} \frac{\sum_{\{i,j\} \in E} |y_i + y_j|}{d \cdot \sum_i |y_i|}$$

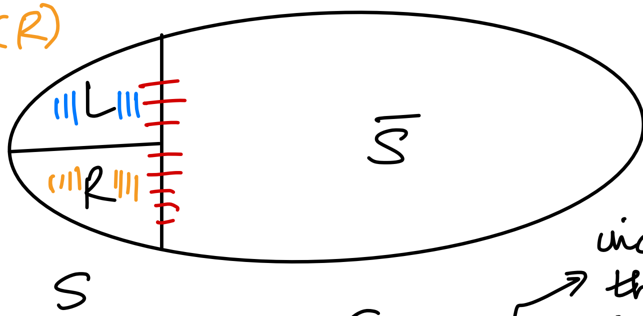
$S = \{ i \mid y_i = \pm 1 \}$, $L = \{ i \mid y_i = +1 \}$

$R = \{ i \mid y_i = -1 \}$

red = $E(S,\bar{S})$

orange = $E(R)$

blue = $E(L)$




frac of edges incident on $S$ that are not in cut$(L,R)$.

$$\beta(G) = \min_{S \subseteq V} \underbrace{\left( \frac{2|E(L)| + 2|E(R)| + |E(S,\bar{S})|}{|S| \cdot d} \right)}_{\beta(S)}$$

If $G$ has a bipartite component, then bipartiteness gap $\beta(G) = 0$.

Given prop, we may just hope to relate $\beta(G)$ with $\lambda_n$.

<u>Def:</u> $\beta_n = 2 - (\lambda_n / d)$.

Theorem [ Trevisan'09, ..., Soto'14]

$$\frac{1}{2}\beta_n \leq \beta(G) \leq \sqrt{2\beta_n}$$

<u>Proof:</u> Let $L'_G$ be such that
$$x^T L'_G x = \sum_{\{i,j\}\in E} (x_i+x_j)^2$$

For any $x$,

$$\sum_{\{i,j\}\in E}\left[(x_i+x_j)^2+(x_i-x_j)^2\right]$$
$$= \sum_{\{i,j\}\in E} 2(x_i^2+x_j^2)$$
$$= 2d\cdot\|x\|_2^2$$

So $x^T L_G x + x^T L'_G x = 2d\cdot\|x\|_2^2$

Thus, since $\lambda_n = \max\limits_{x:\, \|x\|_2 \neq 0} \dfrac{x^T L_G x}{\|x\|_2^2}$

$$\min_{x:\, \|x\|_2 \neq 0} \frac{x^T L'_G x}{\|x\|_2^2} \geq 2d - \lambda_n .$$

Let $S_* = L \cup R$ minimize $\beta(S)$

Let $y_i = \begin{cases} +1 & \text{if } i \in L \\ -1 & \text{if } i \in R \\ 0 & \text{o/w} \end{cases}$

$$2d - \lambda_n \leq \frac{\sum\limits_{\{i,j\} \in E} (y_i + y_j)^2}{\|y\|_2^2} \leq \frac{\sum\limits_{\{i,j\} \in E} 2 \cdot |y_i + y_j|}{\sum |y_i|}$$

$$\leq 2\beta(S_*)\, d$$
$$= 2\beta(G) \cdot d .$$

Divide by $d$ throughout to get the first inequality

Let's now look at the 2nd inequality

Let $y \in \mathbb{R}^n$ be such that

$$y^T L'_G y = (2d - \lambda_n) \|y\|_2^2$$

by scaling, we can assume that

$$\max_i y_i^2 = 1.$$

Rounding: 1) Choose $t \in [0,1]$ uniformly at random

2) Set $x_i = \begin{cases} +1 & \text{if } y_i \geq \sqrt{t} \\ -1 & \text{if } y_i \leq -\sqrt{t} \\ 0 & \text{o/w} \end{cases}$

Notice the contrast with Cheeger rounding.

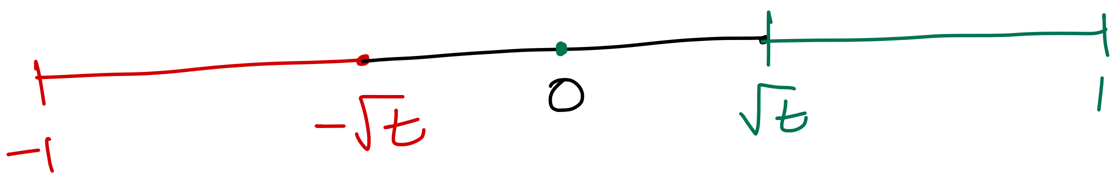
# Analysis of Rounding:

**Claim:** For every $\{i,j\} \in E$,

$$\mathbb{E}|x_i+x_j| \le |y_i+y_j| \quad |y_i|+|y_j|$$

**Proof:** Assume WLOG $y_i^2 \ge y_j^2$

**Case 1:** $y_i, y_j$ have same sign.




If $y_i \ge y_j \ge 0$: possible roundings $(1,0)$ or $(1,1)$.

If $y_i \le y_j \le 0$: possible roundings: $(-1,0)$ or $(-1,-1)$

Either way:

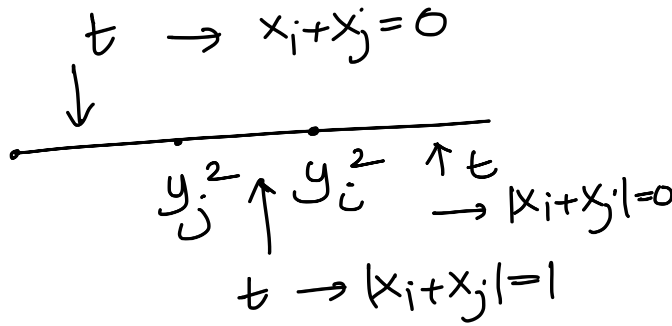
$$\mathbb{E}|x_i+x_j| = 1 \cdot \Pr[y_i^2 \ge t \ge y_j^2]$$

$$+ \quad 2 \cdot \Pr[t \leq y_j^2].$$

$$= (y_i^2 - y_j^2) + 2 \cdot y_j^2 = y_i^2 + y_j^2$$

$$\leq |y_i + y_j|(|y_i| + |y_j|).$$

<u>Case 2:</u> $y_i, y_j$ have opposite signs




$$\mathbb{E}[|x_i + x_j|] = 1 \cdot \Pr[y_i^2 \geq t \geq y_j^2]$$

$$= y_i^2 - y_j^2$$

$$= (y_i + y_j)(y_i - y_j)$$

$$\leq |y_i + y_j|(|y_i| + |y_j|).$$

Summing up the above upper bound over all edges:

$$\sum_{\{i,j\}\in E} \mathbb{E}\,|x_i + x_j|$$

$$\leq \sum_{\{i,j\}\in E} |y_i + y_j|\,(|y_i| + |y_j|).$$

$$\leq \sqrt{\sum_{\{i,j\}\in E} |y_i + y_j|^2} \cdot \sqrt{\sum_{\{i,j\}\in E} (|y_i| + |y_j|)^2}$$

$$\downarrow_{\text{II}} \qquad \downarrow_{\text{IA}}$$

$$\leq \sqrt{(2d-\lambda_n)\|y\|_2^2} \cdot \sqrt{\sum_{\{i,j\}\in E} 2(y_i^2 + y_j^2)}$$

$$= \sqrt{2d-\lambda_n}\,\sqrt{\|y\|_2^2} \cdot \sqrt{2d\|y\|_2^2}$$

$$= \sqrt{2}\,\sqrt{2 - \frac{\lambda_n}{d}} \cdot d\,\|y\|_2^2. \qquad \text{———} \quad (*)$$

And,

$$\mathbb{E} \sum_i |x_i| = \sum_i \Pr[t \le y_i^2]$$
$$= \sum_i y_i^2 = \|y\|_2^2. \quad \text{—} (**)$$

S, $(*)$ & $(**)$ $\Rightarrow$

$$\sum_{\{i,j\} \in E} \mathbb{E}|x_i + x_j| \le \sqrt{2 - \frac{\lambda_n}{d}} \sqrt{2}d \; \mathbb{E} \sum_i |x_i|.$$

$\Rightarrow \exists \, x \in \{\pm 1, 0\}^n$ such that
(via a prop. below)

$$\sum_{\{i,j\} \in E} |x_i + x_j| \le \sqrt{2 - \frac{\lambda_n}{d}} \sqrt{2}d \sum_i |x_i|$$

Rounded distⁿ is supported on at most $n$ possible explicit sets associated to $y$. So can find $x$ just like Cheeger...

Prop: Suppose $\{u_1, u_2, \ldots, u_k\}$
$\{v_1, \ldots, v_k\}$
$\{w_1, \ldots, w_k\}$
are non-neg, $\sum w_i = 1$.

then $$\min_i \frac{u_i}{v_i} \leq \frac{\sum w_i \cdot u_i}{\sum w_i \cdot v_i} = \text{avg}$$

Proof: Suppose not. then,
$$u_i > \text{avg} \cdot v_i \quad \forall i$$
$$\Rightarrow \sum w_i u_i > \text{avg} \cdot \sum w_i v_i$$
or $$\frac{\sum w_i u_i}{\sum w_i v_i} > \text{avg}$$ — Contradiction

Generalization to irregular graphs

$$\mathcal{L}_n = D^{-1/2} \cdot L_G \cdot D^{-1/2}$$

normalized Laplacian

$$D^{-1/2} = \begin{pmatrix} \frac{1}{\sqrt{d_1}} & & & 0 \\ & \frac{1}{\sqrt{d_2}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{d_n}} \end{pmatrix}$$

$d_i$ = degree of vertex $i$.

<u>Def ($\lambda_n$):</u>

$$\lambda_n \mathcal{L}_n := \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{x^T \mathcal{L}_n x}{\|x\|_2^2}$$

$$= \max_{x : x \neq 0} \frac{x^T \cdot D^{-1/2} \cdot L_G \cdot D^{-1/2} \cdot x}{\|x\|_2^2}$$

# Generalization to Irregular graphs

<u>Key change</u>: measure sizes of sets differently, length of vectors diff, use a normalized Laplacian

Let $D^{-1/2} = \begin{pmatrix} \frac{1}{\sqrt{d_1}} & & 0 \\ & \frac{1}{\sqrt{d_2}} & \\ 0 & & \frac{1}{\sqrt{d_n}} \end{pmatrix}$ → $n \times n$ matrix

$d_i = \deg$ of vertex $i$

<u>Def</u> (Normalized Laplacian)

$$\mathcal{L}_n = D^{-1/2} \cdot \mathcal{L}_G \cdot D^{-1/2}.$$

<u>Def</u>. $\max\limits_{x : x \neq 0} \dfrac{x^T \mathcal{L}_n x}{\|x\|_2^2} \stackrel{\text{def}}{=} \lambda_n$

<u>Then</u>, $\lambda_n = \dfrac{x^T D^{-1/2} \cdot \mathcal{L}_G \cdot D^{-1/2} \cdot x}{\|x\|_2^2}$

Setting $[y = D^{-1/2} \cdot x] = \max_{y : y \neq 0} \dfrac{y^T L_G \cdot y}{\sum_i d_i \cdot y_i^2} \overset{\text{def}}{=} \|y\|_D^2$ $\quad y^T D y$

So moving to normalized Laplacian simply changes the "$\ell_2$" length of x to a "weighted $\ell_2$" length.

Def (Bipartiteness Gap)

$$\beta(G) = \min_{\substack{S \subseteq V \\ L \cup R = S \\ L \cap R = \emptyset}} \frac{2|E(L)| + 2|E(R)| + |E(S,\bar{S})|}{\left(\sum_{i \in S} d_i\right)} \qquad \left(\sum_{i \in S} d_i\right) \overset{\text{def}}{=} |S|_D$$

$$= \min_{y \in \{\pm 1, 0\}^n} \frac{\sum_{\{i,j\} \in E} |y_i + y_j|}{\sum_{i \in V} d(i) |y_i|}$$

Observation:

If $G$ has a bip component, then

$\lambda_n(\mathcal{L}_n) = 2$.

Proof: Let $S = L \cup R$ be the bip. connected component.

Let $y_i = \begin{cases} +1 & \text{if } i \in L \\ -1 & \text{if } i \in R \\ 0 & \text{o/w} \end{cases} \quad \forall i$

Then,
$$\frac{y^T \mathcal{L}_G y}{y^T D y} = \frac{\sum_{\{i,j\} \in E} (y_i - y_j)^2}{\sum_i d_i \cdot y_i^2}$$
$$= \frac{4 \cdot |E(L,R)|}{|S|_D} = 2$$
$\square$

Since $S$ is bip, conn. component $|S|_D = 2|E(L,R)|$

Def ($\beta_n$)

$$\beta_n = 2 - \lambda_n .$$

Prop:

$$\beta_n = \min_{y \in \mathbb{R}^n} \frac{\sum_{\{i,j\} \in E} (y_i + y_j)^2}{\sum_i d_i \cdot y_i^2}$$

Proof:

$$\sum_{\{i,j\} \in E} \left[ (y_i + y_j)^2 + (y_i - y_j)^2 \right]$$

$$= 2 \sum_{\{i,j\} \in E} (y_i^2 + y_j^2)$$

$$= 2 \cdot \sum_i d_i \cdot y_i^2 \quad \text{— (1)}$$

$$\& \ \lambda_n \cdot \left( \sum_i d_i y_i^2 \right) = \sum_{\{i,j\} \in E} (y_i - y_j)^2 \quad \text{— (2)}$$

Subtract ② from ① to get Prop.

**Thm [Trevisan]**

$$\underbrace{\frac{1}{2}\beta_n \leq \beta(G)}_{①} \underbrace{\leq \sqrt{2\beta_n}}_{②}$$

**Proof:**

For ①

$$\beta_n = \min_{\substack{y \in \mathbb{R}^n \\ y \neq 0}} \frac{\sum_{\{i,j\} \in E} (y_i + y_j)^2}{\sum_i d_i \cdot y_i^2}$$

$$\leq \min_{y \in \{\pm 1, 0\}^n} \frac{\sum_{\{i,j\} \in E} (y_i + y_j)^2}{\sum_i d_i \cdot y_i^2}$$

$$\leq \min_{y \in \{\pm 1, 0\}^n} \frac{\sum_{\{i,j\} \in E} 2|y_i + y_j|}{\sum_i d_i y_i^2} = 2 \cdot \beta(G)$$

For ②,

let $y \in \mathbb{R}^n$ be such that

$$\lambda_n = \frac{y^T \mathcal{L}_n y}{\|y\|_2^2}.$$

rescale if needed to assure that

$$\max_i y_i^2 = 1.$$

Choose $t \in [0,1)$ uniformly at random.

Set $$X_i = \begin{cases} 1 & \text{if } x_i \geq \sqrt{t} \\ -1 & \text{if } x_i \leq -\sqrt{t} \\ 0 & \text{o/w} \end{cases}$$

Prop: $\mathbb{E}|X_i + X_j| \leq |y_i + y_j| (|y_i| + |y_j|)$

$\forall \{i, j\}$.

Proof: exact same as in regular case.

Summing up over edges.

$$\mathbb{E} \sum_{\{i,j\}\in E} |x_i + x_j|$$

$$\leq \sum_{\{i,j\}\in E} |y_i + y_j| \, (|y_i| + |y_j|)$$

$$\leq \sqrt{\sum_{\{i,j\}\in E} (y_i + y_j)^2} \sqrt{\sum_{\{i,j\}\in E} (|y_i| + |y_j|)^2}$$

$$\leq \sqrt{\beta n \sum_i d_i y_i^2} \sqrt{\sum_{\{i,j\}\in E} 2(y_i^2 + y_j^2)}$$

$$\leq \sqrt{2\beta n} \cdot \sum_i d_i \cdot y_i^2$$

$$\mathbb{E} \sum_i |x_i| d_i = \sum_i \Pr[t \le y_i^2] d_i$$

$$= \sum_i d_i y_i^2$$

<u>So</u>.

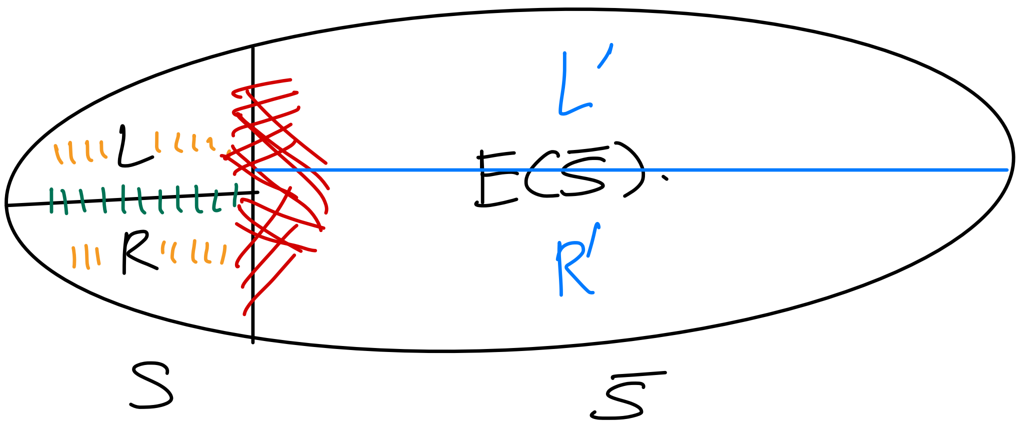
$$\mathbb{E} \sum_{\{i,j\} \in E} |x_i + x_j| \le \sqrt{2\beta n} \cdot \mathbb{E} \sum_i |x_i| \cdot d_i$$

$\Rightarrow \exists$ an $x$ in the support where the above inequality holds. $\square$

[Can find by searching over the $n$ diff threshold cuts]

# Algorithm from Trevisan's inequality

**Idea**: 1) Use Trevisan's inequality to get a "partial rounding"
2) recurse.




**Algorithm**:

1) Apply Trevisan rounding to $G$ to get $S = L \cup R$
2) Apply Trevisan Rounding to $E(\overline{S})$

to get $L'UR'$.

3) return best of $(LUL', RUR')$ & $(LUR', RUL')$.

<u>Theorem:</u> For any $G$,

Let $Alg(G)$ = size of cut output by Algo above.

Then if $\text{max-cut}(G) = 1-\varepsilon$ then $Alg(G) \geq 1 - 4\sqrt{\varepsilon}$.

+ Best of greedy & Trevisan cut yields a $\frac{32}{63}$ approx. to max-cut.

<u>Analysis</u>.

<u>Claim 1:</u> If $\text{maxcut}(G) = 1-\varepsilon$

then $\beta(G) \leq 2\varepsilon$

<u>Proof:</u> Let $S_*$ be an optimal max-cut in $G$

Choose $S = V$

$L = S_*$

$R = \bar{S}_*$

<u>Then</u>, $\beta(S) \leq \dfrac{2|E(S_*)| + 2|E(\bar{S}_*)| + |E(S,\bar{S})|}{d \cdot |S|}$

$=0$

<u>Note</u>: $|E(S_*)| + |E(\bar{S}_*)| + |E(S_*, \bar{S}_*)| = m$

<u>So</u>. $|E(S_*)| + |E(\bar{S}_*)| < m \cdot \varepsilon$.

So $\beta(S_*) \leq \dfrac{2 \cdot \varepsilon \cdot m}{d n} = \varepsilon.$

From Trevisan's inequality,

$$\frac{1}{2}\beta_n \leq \beta(G) \leq \beta(S_*)$$

Thus, $\beta_n \leq 2\varepsilon$

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Thus,

applying Trevisan's Rounding to $G$

a set $S$, partitioned into $L \cup R$

s.t.

$$\beta(S) = \frac{2|E(L)| + 2|E(R)| + |E(S,\bar{S})|}{d \cdot |S|}$$

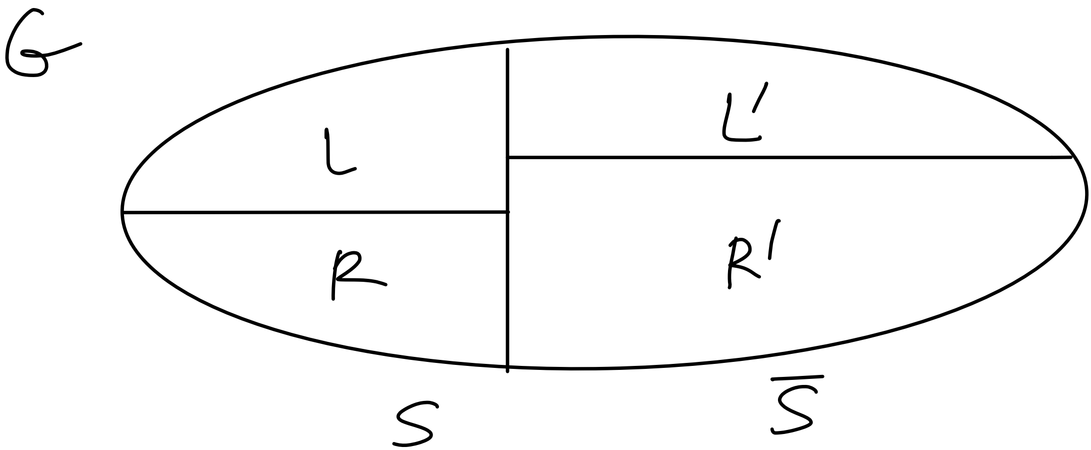
$$\leq \sqrt{2 \cdot 2\varepsilon} = 2\sqrt{\varepsilon}.$$

$\boxed{\text{Idea:}}$ beat $\frac{1}{2}$ in the $E(L,R)$ cut

& $E(L',R')$ cut

do "OK" on $E(S,\bar{S})$ (i.e. cut $\frac{1}{2}$ edges)

recurse on $E(\bar{S})$




$$\text{max-cut}(G) \leq \text{max-cut}(E(\bar{S})) + |E(L)| + |E(R)| + |E(L,R)| + |E(S,\bar{S})|$$

How does the better of $(LUL', RUR')$ & $(LUR', RUL')$ do?

<u>Observation (merging is easy)</u>

$$|E(S,\bar{S})| = |E(L,L')| + |E(L,R')| + |E(R,L')| + |E(R,R')|$$

<u>So</u>,

$$\max\{|E(L,L')| + |E(R,R')|, |E(L,R')| + |E(R,L')|\} \geq \frac{1}{2}|E(S,\bar{S})|$$

Thus, better of $(LUL', RUR')$ & $(LUR', RUL')$ cuts $\geq \frac{1}{2}|E(S,\bar{S})|$

edges from the cut $E(S,\bar{S})$.

Thus,

$$\mathrm{Alg}(G) \geq |E(L,R)| + \frac{1}{2}|E(S,\bar{S})| + \mathrm{Alg}(E(\bar{S})).$$

Lemma: If $\beta(S) \leq 2\sqrt{\varepsilon}$, then

$$\frac{|E(L,R)|}{|E(L)|+|E(R)| + |E(L,R)|+|E(S,\bar{S})|} \geq 1-2\sqrt{\varepsilon}.$$

$$\Rightarrow 2\sqrt{\varepsilon} \geq \beta(S) = \frac{2|E(L)|+2|E(R)|+|E(S,\bar{S})|}{d\cdot|S|}$$

$d(S)$

$$= 2|E(L)| + 2|E(R)| + 2|E(L,R)| + |E(S,\bar{S})|$$

$f(S) \leq 2\sqrt{\varepsilon}$ means that

$$\frac{2|E(L)| + 2|E(R)| + |E(S,\bar{S})|}{2|E(L)| + 2|E(R)| + 2|E(L,R)| + |E(S,\bar{S})|} \leq 2\sqrt{\varepsilon}$$

$$\Rightarrow 1 - \frac{2|E(L,R)|}{2|E(L)| + 2|E(R)| + 2|E(L,R)| + |E(S,\bar{S})|} \leq 2\sqrt{\varepsilon}.$$

or

$$\frac{|E(L,R)|}{|E(L)|+|E(R)|+|E(L,R)| + \frac{1}{2}|E(S,\bar{S})|} \geq 1-2\sqrt{\varepsilon}$$

add $\frac{1}{2}|E(S,\bar{S})|$ to both numerator & denominator only increases the ratio

So

$$\frac{|E(L,R)|+\frac{1}{2}|E(S,\bar{S})|}{|E(L)|+|E(R)|+|E(L,R)|+|E(S,\bar{S})|} \geq 1-2\sqrt{\varepsilon}$$

$\square$

So,

$$\mathrm{Alg}(G) \geq (1-2\sqrt{\varepsilon})\left(\frac{|E(L,R)|+\frac{1}{2}|E(S,\bar{S})|}{|E(S)|+|E(S,\bar{S})|}\right) + \mathrm{Alg}(E(\bar{S})). \quad ***$$

How much can max-cut$(E(\bar{S}))$ be?

Prop: $\text{max-cut}(E(\bar{S})) \geq 1 - \varepsilon \frac{m}{E(\bar{S}}$

Proof: $(1-\varepsilon)m = (\text{max-cut}(G))\cdot m$

$\leq |E(S)| + |E(S,\bar{S})| + (\text{max-cut}(E(\bar{S})))|E(\bar{S})|$

So: $\text{max-cut}(E(\bar{S}))$

$$\geq \frac{1}{|E(\bar{S})|}\left[(1-\varepsilon)\left[|E(S)|+|E(S,\bar{S})|+|E(\bar{S})|\right] - |E(S)| - |E(S,\bar{S})|\right]$$

$$\geq \frac{1}{|E(\bar{S})|} \cdot \Big[ (1-\varepsilon) \cdot |E(\bar{S})| - \varepsilon \cdot \big[ |E(S)| + |E(S,\bar{S})| \big] \Big]$$

$$= 1-\varepsilon - \varepsilon \cdot \left( \frac{|E(S)| + |E(S,\bar{S})|}{|E(\bar{S})|} \right)$$

$$= 1 - \varepsilon \cdot \frac{m}{|E(\bar{S})|}$$

<u>Proof of Theorem</u>:

Let $V(\varepsilon) = \min_{\substack{G:\ \text{max-cut}(G) = 1-\varepsilon \\ G \text{ on } n \text{ verts}}} 1 - Alg(G)$.

Suppose $\frac{|E(\bar{S})|}{m} = \delta$. Then,

$$V(\varepsilon) \lesssim 2\sqrt{\varepsilon}\,(1-\delta) + \delta \cdot V\!\left(\frac{\varepsilon}{\delta}\right).$$

Inductively assume $V(\varepsilon) \leq 4\sqrt{\varepsilon}$ for graphs $n < n$ vertices.

$$V(\varepsilon) \leq 2\sqrt{\varepsilon}\,(1-\delta) + \delta\, 4 \sqrt{\frac{\varepsilon}{\delta}}$$

$$= 2\sqrt{\varepsilon}\,(1-\delta) + \sqrt{\delta}\, 4\sqrt{\varepsilon}.$$

Now $2\sqrt{\varepsilon}\,(1-\delta)$

$$= 2\sqrt{\varepsilon}\,\underbrace{(1+\sqrt{\delta})}_{\leq 2} \cdot (1-\sqrt{\delta}).$$

$$\leq 4\sqrt{\varepsilon}\,(1-\sqrt{\delta}).$$

Thus $V(\varepsilon) \leq 4\sqrt{\varepsilon}$ as desired.

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<u>Approx. Ratio:</u>

$$\max\left\{\frac{1}{2(1-\varepsilon)},\ \frac{1-4\sqrt{\varepsilon}}{1-\varepsilon}\right\}$$

equal at $\varepsilon = \frac{1}{64}$ at which each is $\frac{32}{63}$