Lec 5

GAPS FOR MAX-CUT

Last time: the GW algorithm.

QP:

$$\max \frac{1}{4m} \sum_{\{u,v\} \in E} (x_u - x_v)^2$$

$$= \frac{1}{2} - \frac{1}{2m} \sum_{\{u,v\} \in E} x_u \cdot x_v$$

$$= \frac{1}{2} - \frac{1}{2} \mathbb{E}_{\{u,v\} \in E}\, x_u x_v$$

s.t. $x_i \in \{\pm 1\}\ \forall i$

$\xrightarrow[\text{relax}]{\text{SDP}}$

SDP:

$$\max \frac{1}{2} - \frac{1}{2m} \sum_{\{u,v\} \in E} X_{u,v}$$

$$= \frac{1}{2} - \frac{1}{2} \mathbb{E}_{\{u,v\} \in E}\, X_{u,v}$$

$$X \succcurlyeq 0$$

$$\mathrm{diag}(X) = \vec{1}$$

Rounding: 1) $\underset{\in \mathbb{R}^{n \times n}}{X} = \underset{\in \mathbb{R}^{n \times n}}{Z} \cdot Z^T$ (Cholesky Factorization)

$$= \begin{array}{cc} & v \\ u & \begin{pmatrix} \\ \langle Z_u, Z_v \rangle \\ \\ \end{pmatrix} \end{array}$$

$\{Z_u\}_{u \in \mathbb{R}^n}$ ↓ n-dim vectors of unit length.

2) Choose $g = (g_1, \ldots, g_n) \sim$ std. gaussian vector

3) Set $x_u = \mathrm{sign}(\langle g, Z_u \rangle)$ for all $1 \le u \le n$.

Analysis: For every $1 \le u, v \le n$, if $\frac{1}{2} - \frac{1}{2} X_{u,v} = 1 - \eta$

then, $\mathbb{E}_x \left(\frac{1}{2} - \frac{1}{2} x_u \cdot x_v\right) \geq 1 - 2 \cdot \sqrt{\eta}$.

Equivalently,

SDP assigns vectors of unit length to each vertex in V.

$$u \longrightarrow \vec{u}.$$

$$\text{SDP value} = \frac{1}{2} - \frac{1}{2m} \sum_{\{u,v\} \in E} \langle \vec{u}, \vec{v} \rangle$$

$$= \frac{1}{2} - \frac{1}{2} \mathop{\mathbb{E}}_{\{u,v\} \sim E} \langle \vec{u}, \vec{v} \rangle$$

uniformly random edge

General helpful tip:

Set $\longleftrightarrow$ uniform distribution on the set.

$$\overset{\text{SDP value}}{\overset{\shortparallel}{\boxed{SDP(G)}}} = \frac{1}{2} - \frac{1}{2}\,\mathbb{E}_{\{u,v\}\in E}\langle \vec{u}, \vec{v}\rangle$$

<u>Recall</u>: If $\langle \vec{u}, \vec{v}\rangle = \rho$, then

SDP: $\frac{1}{2} - \frac{1}{2}\langle \vec{u}, \vec{v}\rangle = \frac{1}{2} - \frac{1}{2}\rho$

rounding: $\mathbb{E}_{x}\left[\frac{1}{2} - \frac{1}{2} x_u \cdot x_v\right] = \dfrac{\text{arc-cos}(\rho)}{\pi}$

rounded $\{\pm 1\}$-coord. vector

ρ_* : minimizer of $\dfrac{\frac{\arccos(\rho)}{\pi}}{\frac{1}{2} - \frac{1}{2}\rho}$

$\rho_* \approx -0.69$, min-value $\alpha_{GW} \approx 0.878\ldots$

Motivating question: Is there a better algorithm?

Is there a better rounding of the same SDP?

Exolore via "gaps".

<u>Integrality Gap</u> : "how far" can the $SDP(G)$ objective value be from max-cut(G)?

<u>Lemma</u> (Qualitative Integrality Gap)

For every $k \in \mathbb{N}$, there's a graph C_{2k+1} on $2k$ vertices such that

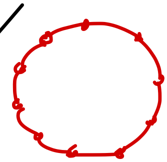
$$SDP(C_{2k+1}) \geq 1 - O\left(\frac{1}{k^2}\right).$$

$k \sim 1/\sqrt{\varepsilon}$
max-cut $\sim 1-\sqrt{\varepsilon}$
SDP $\sim 1-O(\varepsilon)$

$$\text{but} \quad \text{max-cut}(C_{2k+1}) \leq 1 - \frac{1}{2k+1}$$

<u>Proof:</u> graph $=$ $(2k+1)$-length cycle.

$$\text{max-cut}(C_{2k+1}) = 1 - \frac{1}{2k+1} \checkmark$$



C_{11}

How do we prove that $SDP(C_{2k+1})$ is $1-O(\frac{1}{k^2})$?

must produce feasible sol^n $\equiv$ unit vector assignment to vertices.

$$\Longleftrightarrow \quad \mathbb{E}_{\{u,v\}\in E} \langle \vec{u}, \vec{v} \rangle \leq -\left(1 - O\left(\tfrac{1}{k^2}\right)\right).$$

$$\Longleftrightarrow \quad \mathbb{E}_{\{u,v\}\in E} \langle \vec{u}, \vec{v} \rangle \leq -\left(1 - O\left(\tfrac{1}{k^2}\right)\right).$$

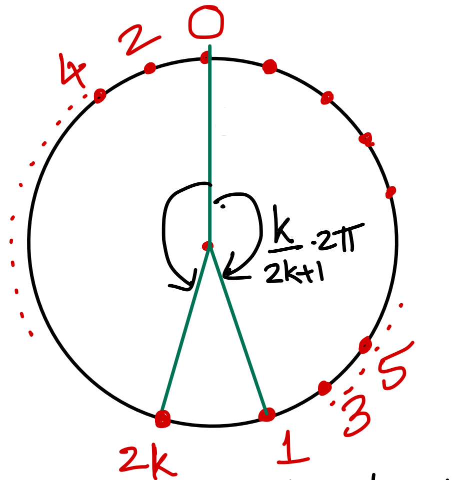
must place end points of edges at as large an obtuse angle as possible.

Idea:

the pie is divided into equiangular, odd # $2k+1$ of equal size sectors.

the construction ensures that neighbors are at an angle of $k\,\theta_{k+1} = \dfrac{2k\pi}{2k+1}$

$= \pi - \dfrac{\pi}{2k+1}$



"2D unit sphere" = unit circle in the plane.

vertex $j \longrightarrow \left[(j\cdot k) \bmod 2k+1\right] \cdot \dfrac{2\pi}{2k+1}$ angle

$j \leftrightarrow (j\cdot k) \bmod 2k+1$ is a bijection on $\{0,1,\ldots,2k\}$.

What's the SDP value?

End point of all edges are at an angle of $\left(\pi - \frac{\pi}{k+1}\right)$.

$$\mathop{\mathbb{E}}_{\{u,v\}\in E} \langle \vec{u}, \vec{v} \rangle = \cos\left(\pi - \frac{\pi}{k+1}\right)$$

$$= -\cos\left(\frac{\pi}{k+1}\right).$$

$$= -\sqrt{1 - \sin^2\left(\frac{\pi}{k+1}\right)}$$

$$\leq -\sqrt{1 - \left(\frac{\pi}{k+1} - O\left(\frac{\pi^3}{(k+1)^3}\right)\right)^2}$$

$$\sim -\left(1 - O\left(\frac{1}{k^2}\right)\right).$$

$\square$

The above example gives an integrality gap that matches the qualitative approx. curve of GW.

Tighter example?

Cycle ~ embeddable on 2D unit sphere

Feige-Schechtman '01: A higher dim generalization of the 2d construction gives the exact $\alpha_{GW} - \varepsilon$ integrality gap for every $\varepsilon > 0$.

Theorem [Feige-Schechtman]: ~~For~~ For every $\varepsilon > 0$, there is a graph G such that

$$\text{max-cut}(G) \leq (\alpha_{GW} + \varepsilon) \cdot SDP(G).$$

We will sketch the construction here.

Preliminary considerations:

Suppose we construct $G(V, E)$ with an embedding $V \to S^{d-1}$ ⤳ unit d-dim sphere

$u \to z_u$

s.t. $\{u, v\} \in E \Rightarrow \langle \vec{u}, \vec{v} \rangle \approx \rho$

Then $SDP(G) \approx \frac{1}{2} - \frac{1}{2}\rho$.

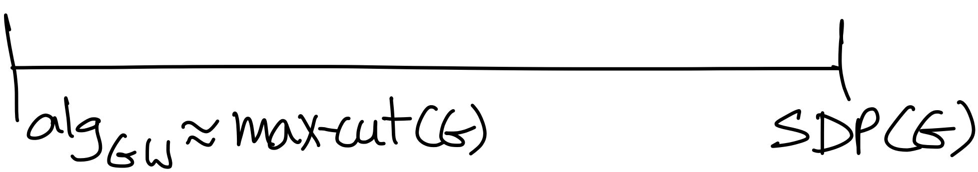
For integrality gap to be α_{GW} $\sim 0.878\ldots$,

we need $\text{max-cut}(G) \leq \alpha_{GW}\left(\frac{1}{2} - \frac{1}{2}\rho\right)$.

But $\text{max-cut}(G) \geq \text{alg}_{GW} \geq \dfrac{\arccos(\rho)}{\pi}$

<u>So</u>: we must have $\rho \approx \rho_*$ and

$\text{alg}_{GW} \approx \text{max-cut}(G)$.



We will define a graph directly by giving an "embedding" on to the unit d-dim sphere.

<u>Def (Embedded graph)</u>

An embedded graph $G(V,E)$ is a weighted graph such that $V \subseteq$ d-dim unit sphere
$u \longrightarrow z_u$
for some d.

Let $\mathrm{Obj}(G) = \mathop{\mathbb{E}}_{\{u,v\} \in E} w_{u,v} \cdot \left(\frac{1}{2} - \frac{1}{2} \langle \vec{u}, \vec{v} \rangle \right)$

<u>Prop</u>: $\mathrm{Obj}(G) \leq \mathrm{SDP}(G)$.

<u>Proof</u>: use z_us to get feasible SDP solution of value $= \mathrm{Obj}(G)$. ✓

To ensure $\mathrm{Obj}(G) \sim \frac{1}{2} - \frac{1}{2} \rho_*$,

must ensure $\langle \vec{u}, \vec{v} \rangle \sim \rho_* \ \forall \{u,v\} \in E$.

We'll construct an "infinite geometric graph" and then discretize it.

Vertices $= S^{d-1}$:
all possible points on the surface of the unit sphere.

Edges $\sim$ prob distⁿ over pairs of unit vectors.

will be symmetric (i.e. prob of (w_1, w_2) = prob of (w_2, w_1)).

Then, $\mathrm{Obj}(G) = \underset{\substack{\{u,v\} \\ \sim E}}{\mathbb{E}}\left[\frac{1}{2} - \frac{1}{2}\langle \vec{u}, \vec{v} \rangle\right]$

What about max-cut(G) ?

Every cut is a subset of vertices.

$\equiv \pm 1$ indicator

$\equiv f: S^{d-1} \to \{\pm 1\}$.

Thus,

$$\text{max-cut}(G) = \max_{f: V \to \{\pm 1\}} \Pr_{\{u,v\} \sim E} [f(u) \neq f(v)]$$

[overlooking the issue of existence of the max]

<u>Edge Distrubution:</u> 1

uniform draw from E $\equiv$ pick $\vec{u}, \vec{v}$ uniformly conditioned on $\langle \vec{u}, \vec{v} \rangle \leq \rho_*$.

Fact: $\mathbb{E}_{\vec{u},\vec{v} \sim S^{d-1}}\left[\langle \vec{u},\vec{v}\rangle \mid \langle \vec{u},\vec{v}\rangle \leq \rho_*\right]$

$$\geq \rho^* - O\left(\frac{1}{\sqrt{d}}\right).$$

"magic of high dim geometry".

So: we immediately ensure that

$$\mathrm{Obj}(G) \geq \tfrac{1}{2} - \tfrac{1}{2}\rho_* - o_d(1).$$

What about max-cut(G)?

Say optimal cut is $(A, \bar{A})$ for $A \subseteq S^{d-1}$!

"# vertices in A" = fractional surface area of A.
= a (say).

$\mu_{\rho_*}(A)$ = frac of edges crossing A.

$$\mu_{\rho_*}(A) = \Pr_{\{u,v\} \sim E}\left[A(u) \neq A(v) \right]$$

where $A(u) = 1$ iff $u \in A$.

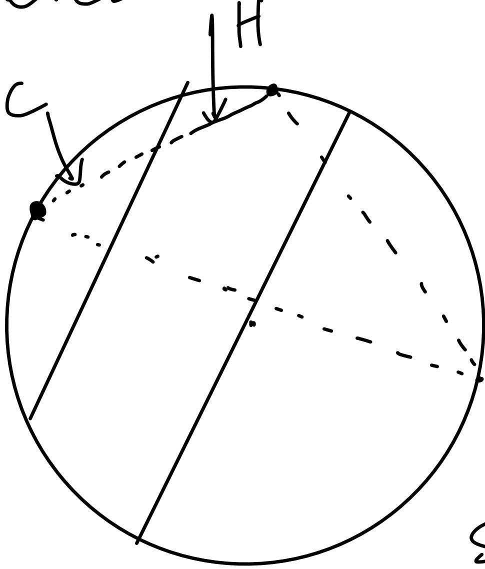
hardest part of proof:

<u>Thm</u>: Fix $0 \le a \le 1$ & $0 \ge \rho \ge -1$.

Then, max of $\mu_\rho(A)$ where A ranges over all measurable subsets of S^{d-1} of frac surf area a is attained for "cap" of surf area a.

<u>Cap</u>: points on one side of some hyperplane cut $\equiv \{\vec{u} \mid \langle \vec{u}, h \rangle \geq \tau\}$

Feige-Schechtman argue in fact that $\mu_\rho(A)$ is maximized at hemispheres.



any point $p \in H/C$ has more edges going out of H than into C. So adding p to C can't hurt.

all hemispheres have same size cuts by symmetry.

Observation: Cut output by GW algo is a hemisphere.

$g : (g_1 \dots g_n)$ ~ std. gaussian vector

$$\underline{\text{cut}} = \{\vec{u} \mid \underbrace{\langle g, \vec{u} \rangle \geq 0}_{\text{hemisphere!}}\}.$$

So GW outputs optimal cuts in the graph → & they have the value ~ $\frac{\text{arc-cos}(\rho)}{\pi}$.

Final construction: vertex = regions of sphere of frac surf area = ε.

This yields an integrality gap of $0.878 + \varepsilon$.

Note: On instances that achieve the integrality gap, GW algo outputs optimal cuts!

So the algorithm actually does as well as it possibly could.

Algorithmic Gap.

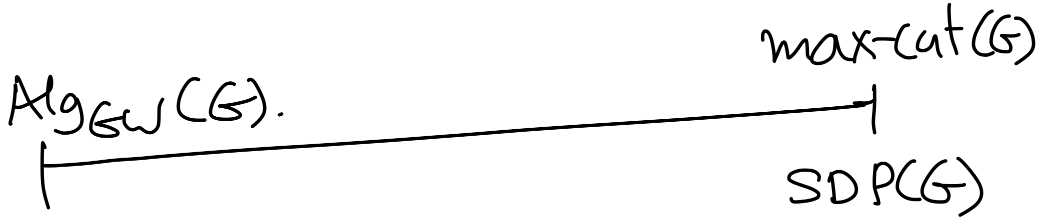
Are there instances G where

$$Alg_{GW} \leq (\alpha_{GW} + \varepsilon) \cdot \text{max-cut}(G) \ ?$$

for arbitrarily tiny $\varepsilon > 0$.

In this case, clearly,

$$SDP(G) \approx \text{max-cut}(G)$$



Alg_{GW} is a function of the SDP solution $\dashrightarrow$ there are multiple SDP solutions of optimal value.

In particular, the <u>integral</u> sol^n!

Clearly, Alg_{GW} cannot be

$$\leq \alpha_{GW} \, \text{max-cut}(G) \text{ if}$$

SDP solution is integral opt

Theorem [Karloff] For all $\varepsilon > 0$, there is a graph G and an optimal SDP solution $\{u \to \vec{u}\}_{u \in V}$ s.t.

$$SDP(G) \leq \text{max-cut}(G) + \varepsilon \quad \&$$

$$alg_{GW}(G) \leq (\alpha_{GW} + \varepsilon) \cdot \text{max-cut}(G).$$

Proof: We will again define the graph together with an (optimal) embedding.

Our graph (embedding) $= \left\{ +\frac{1}{\sqrt{d}}, -\frac{1}{\sqrt{d}} \right\}^d$

"hypercube" with labels normalized

to be of length $= 1$.

As before we must add edges between pairs where the inner product is $\approx \rho_*$.

<u>Edge distn</u>: 1) pick $\vec{u}$ uniform $\{\pm \frac{1}{\sqrt{d}}\}^d$

2) negate each coordinate indep w.p. $\frac{1}{2} - \frac{1}{2}\rho_*$ to get $\vec{v}$.

3) output $(\vec{u}, \vec{v})$.

$$\mathop{\mathbb{E}}_{\vec{u},\vec{v}} \langle \vec{u}, \vec{v} \rangle = \frac{1}{d} \sum_i \mathbb{E}\, \vec{u}_i \cdot \vec{v}_i$$

$$= \frac{1}{d} \sum_i \left(\frac{1}{2} + \frac{1}{2}\rho_*\right) - \left(\frac{1}{2} - \frac{1}{2}\rho_*\right)$$

$$= \rho_* . \checkmark$$

$$\mathrm{Obj}(G) = \mathop{\mathbb{E}}_{\{\vec{u},\vec{v}\}\in E}\left[\frac{1}{2} - \frac{1}{2}\langle \vec{u},\vec{v}\rangle\right]$$

$$= \frac{1}{2} - \frac{1}{2}\mathop{\mathbb{E}}_{\{u,v\}\in E}\langle \vec{u},\vec{v}\rangle$$

$$= \frac{1}{2} - \frac{1}{2}\rho_* $$

<u>Fact</u>: $\displaystyle \Pr_{\{\vec{u},\vec{v}\}\in E}\left[|\langle \vec{u},\vec{v}\rangle - \rho_*| > \frac{t}{\sqrt{d}}\right] \le 2^{-O(t^2)}.$

(Chernoff bound).

<u>So</u> most edges in fact have the corresponding inner product $\sim \rho_*$.

So, GW will output a cut of

value $\sim \dfrac{\text{arc-cos}(\rho^*)}{\pi} + o_d(1)$

$o_d(1) \to 0$ as $d \to \infty$

We will prove, however, that there are cuts of value $\frac{1}{2} - \frac{1}{2}\rho_*$.

Dictator Cuts:

$$D_i = \{\vec{u} \mid \vec{u}_i = \pm \tfrac{1}{\sqrt{d}}\}.$$

What's the size of these cuts?

$$\Pr_{\{\vec{u},\vec{v}\} \in E} [D_i(\vec{u}) \neq D_i(\vec{v})]$$

$= \Pr[i^{th} \text{ coord. of } \vec{u} \text{ is flipped}]$

$= \frac{1}{2} - \frac{1}{2}\rho_*$. ✓

$\square$

Are we done?

We've shown: $\text{Obj}(G) \sim \frac{1}{2} - \frac{1}{2}\rho_*$

$\text{alg}_{GW}(G) \leq \frac{\arccos(\rho_*)}{\pi}$

& $\text{max-cut}(G) \geq \frac{1}{2} - \frac{1}{2}\rho_*$.

Need to show our embedding is optimal. That is, there is no other better SDP solution.

This needs some very basic Fourier analysis that we will see in a later class.

<u>Fact (Dictator is Stablest)</u>:

Let $f : \{\pm 1\}^n \to \mathbb{R}$. Let

$$S_\rho = \mathop{\mathbb{E}}_{(x,y) \sim E} [f(x) \cdot f(y)]$$

Then $S_\rho(f) \geq \rho \cdot \mathop{\mathbb{E}}_x f(x)^2$

How does this help?

Let $Z : \{-1,1\}^n \to S^{d-1}$ be the SDP embedding.

Then, let $F_j(x)$ be the j^{th} coordinate of $Z(x)$ for every $x \in \{\pm 1\}^n$? Then $\sum_{i=1}^{d} F_j(x)^2 = 1$.

Then, $\mathbb{E}_{(x,y)\sim E} F_j(x) \cdot F_j(y) \geq \rho \cdot \mathbb{E}_x F_j(x)^2$

Add from $j=1$ to d & use that $\sum_{i=1}^{d} F_j(x)^2 = 1$. $\forall x$.

Comments:

* Hypercube vs Sphere

rounding seems to behave the same but hypercube has special large dictator cuts. Sphere doesn't.

* Can we strengthen the SDP relaxation so that sphere is no longer an integrality gap?

<u>YES!</u> "degree 4 sum-of-squares"

Does it help improve Goemans Williamson? We do not know.