UNIQUE GAMES CONJECTORE

-Strengthening of P=NP hypothesis -posits hardness of a natural problem - yields optimal hardness regults for Max-Cut, Vertex Cover,.... - trath not yet known - Amazingly vich theory with interplay between Dhardness of approx. 2) algo design va SDPs. 3) Methods from Analysis of Boolean Functions, Geometry - - -

The Robben & the Conjecture Input: equations of the form 2 vars in each equation allmod Goal: Satisfy max frac of them Def C Value): max frac of constraints Batisfrage Random Assignment: 1 Theorem: If value = 1, Can find

a satassignment en polytime. Proof: Solve Lenéar equations via Gaussian elimnation --CONJECTURE [Khot'02] IFEZO, Yp large enough, (1-E, E) - MAX ZLINCP) لع NP-havd. u It is NP-hard to 'find an E sat assignment for a I-E sat instance". Best polytime algo: 1-E, 1-O(JEloyk) dependent Can beat brute-force Search: alphabet size"

Avora-Barak-Steurer 10 $2^{N^{\Sigma}}$ time algo, if $opt = 1-\Sigma$, vound = $1-\Sigma^{Y_{0}}$. "indep of alphabet size"

Lots of interesting work on both algos & lower bounds Most recently-Thm [2018] [2-to-2 Games Thm] $(\frac{1}{2}, \varepsilon)$ -UG is WP-Hard. $\longrightarrow 1-\varepsilon \implies U.G.C.$

Today & next 2 classes) a glempse of this theory. optimiNP We will prove hardness of Max-Cut assuming the U.G.C. We will do it in a way that shows that there's a principled theory of Such optimal hardness reductions. And this throng directly builds hardness reductions from Integrality Gaps for SDP for a large class of problems

FAILUREOFSDP -> UG Hardness-...

Theorem [KhotKindler Mossell O'donnell 04] 4270, dowt 2 approx. Max-cut is NP-Hard assuming the U.G.C. min arc-cos(p) pro T (12-2p) 1 Like the examples we saw en last lecture, this theorem we will prove this theorem via "gadget" reductions "Gadget Reductions", Q: P, -> P2. Pz instance = "local transformation" of modify constant size portions instance for P1. E.g. Reduction for Vertex Cover that replaces each clause by 7 vertices

each edge of the resulting Is instance. depends only on 2 clauses in the injoint 3 SAT instance.

Our gadgets need to get the exact $\frac{1}{2} - \frac{1}{2}g \rightarrow \frac{\operatorname{arc-cos}(e)}{\pi} gap.$ So need some "geometry" to shar

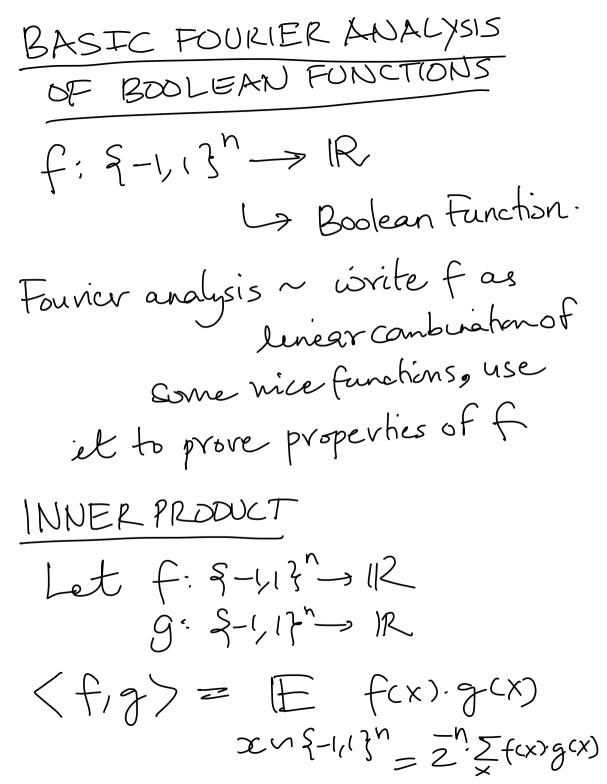
In fact, our gadgets will in a precise sense come from our Integrality Gap & Rounding Gap Instances for Max up. Cut '

Let me venied you of those... Both were "embedded graphs" -> each Vertex came with a vector labeling.t.

Integrality Gap (Feige Schechtman graph) Vertices = (discretization) of write edge : pick û, i wiformly from distribution : S^{d-1} conditioned on くびパフミ ら*. $\frac{\text{SDP OBJ}}{\{\vec{u},\vec{v}\}} : \mathbb{E}\left[\frac{1}{2} - \frac{1}{2} < \vec{u},\vec{v}\right] \\ \frac{1}{\{\vec{u},\vec{v}\}} = \frac{1}{2} - \frac{1}{2}S_{*}.$ Analysis: (1) hemisphere Certs are optimal for this graph D'hemisphere cuts have Value ~ arc-cos (Pre) T Same analysis as our rounding.

Rounding Gap D'Vertices: Corners of hypercube $\xi \pm J = J^d$ scaled down to be cent vectors · i) pick i at vandom 2) edge distⁿ 2) pick V by flipping each Coordinate of it indep W.p. <u>1</u> - <u>1</u> ?* · 3) Ontput { U,V }. (E<u,V) = ?* $SDP OBJ = \frac{1}{2} - \frac{1}{2} q_*$ True Max-Cut:, also 12-29x $D_{i} = \left\{ \vec{u} \mid \vec{u}_{i} = \pm \right\}.$ $P_{v} \left[D_{i}(\vec{u}) \neq D_{i}(\vec{v}) \right] = \frac{1}{2} \frac{P_{*}}{2}$

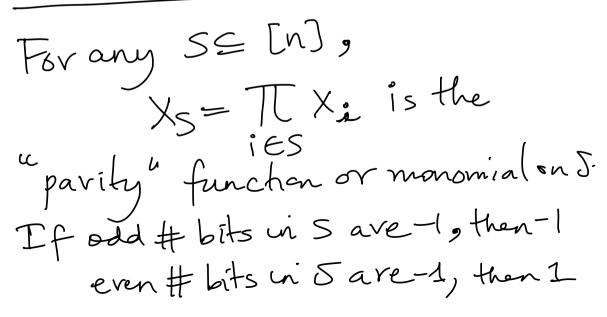
Our gadget for Max-Cuthypercabe graph: to analyze cuts in Such graphs, useful to adopt "funchen view" Every subset of $\leftarrow f: f:f!$ vertices $S \longrightarrow f:f!$ $\downarrow f(x)=tl$ $\downarrow f(x)=tl$ $\rightarrow xc$ $\rightarrow \uparrow \pm \uparrow$ f(x) = +1 $\rightarrow x \in S$ f(x) =-197 xĘS. Value of cuts = Pr[f(x) ≠ f(y)] fx,y]uedge dist Ned machinery to reason about such quantities



"Event fig as vectors of length 2ⁿ, take inner products, rescale by 2ⁿ". Note: EXi=0. Fi X

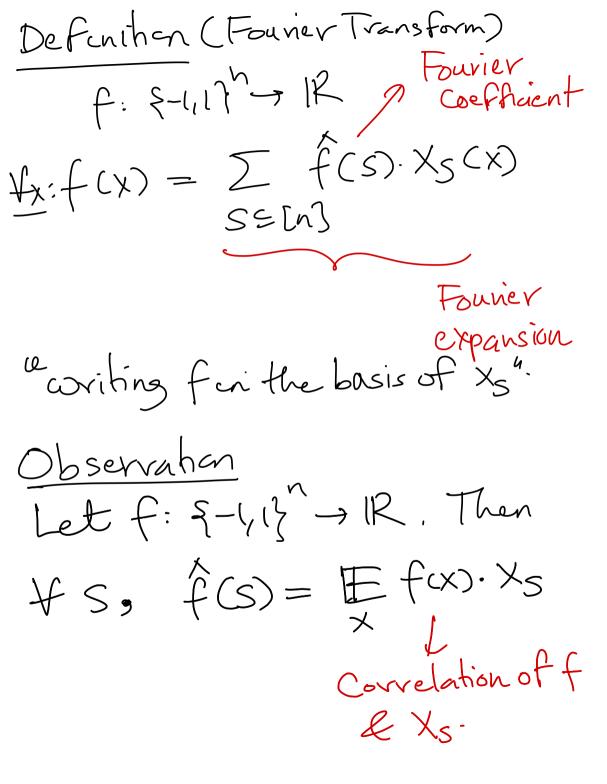
NORM $\|f\|_2^2 = \langle f_i f_i \rangle = \mathop{\mathbb{E}}_{\times} f(x)^2$

Def C Parity functions/Monomials)



 $\frac{66}{E_x} \times S = 0 \quad if S \neq p$ $If S \neq T_{9} \\ \langle X_{S}, X_{T} \rangle = 0$ $TfS=T, \langle X_S, X_S \rangle = ||X_S||_2^2$ = 1Proof: $X_{s'}X_T = TT X_i \cdot TT X_T$ ies iet = TT Xi². TT Xi iesnt iesbt = T X_i IESAT If SAT = D = T = T EXi xiesat iesat

Observation: The set of functions ZXS [SE [n]] form a orthonormal basis w.r.t. the enner product above. Proof. We proved that each Xs has length: 11 Xs 11/2=1 $& \forall s \neq T \langle X_{s}, X_{T} \rangle = 0$ & there are 2ⁿ such functions & the deri of space 5 2ⁿ. Thus each f can be expanded as lin comb. of Xs.



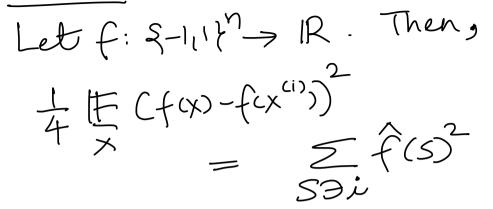
Proof: $f(x) = \Sigma f(S) X_S(x)$ \leq Efcx)-XTCX) $E_{X} = \sum_{s} f(s) \cdot X_{s}(w) \times f(x)$ $= \sum_{S} \widehat{f}(S) \cdot \mathbb{E} X_{S}(X) \cdot X_{T}(X) \times 1/2$ o' 1 if s≠t if s=t $= \hat{f}(T)$ Observation: $f: g-i(1)^n \rightarrow \mathbb{R}$ $\mathbb{E}_{X} f(x)^{2} = \sum_{s} \hat{f}(s)^{2}$ Then,

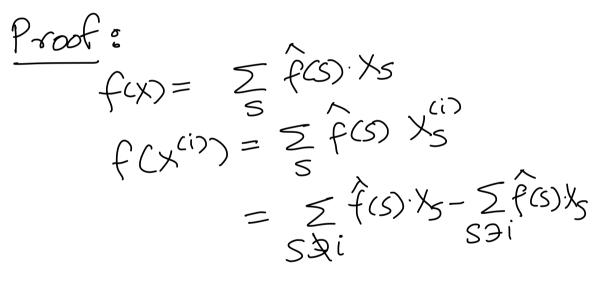
Proof: $f(x) = \sum_{s} f(s) \cdot X_{s}(x)$ $= \sum \hat{f}(S) \cdot \hat{f}(T)$ E XS XT S,T X $= \lesssim f(s)^2.$

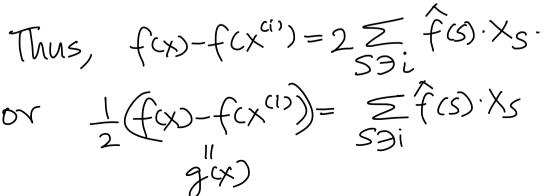
Influence of Functions Inf: (f) : influence of i-th variable on f. Def (for Boolean valued functions) $f: \mathfrak{F} - \mathfrak{f}, \mathfrak{F} \to \mathfrak{F} - \mathfrak{f}, \mathfrak{F}$

" prot that oit a vandom X, flipping it but changes the value of f"

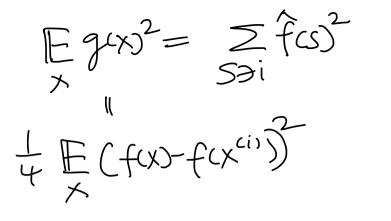
Lemma



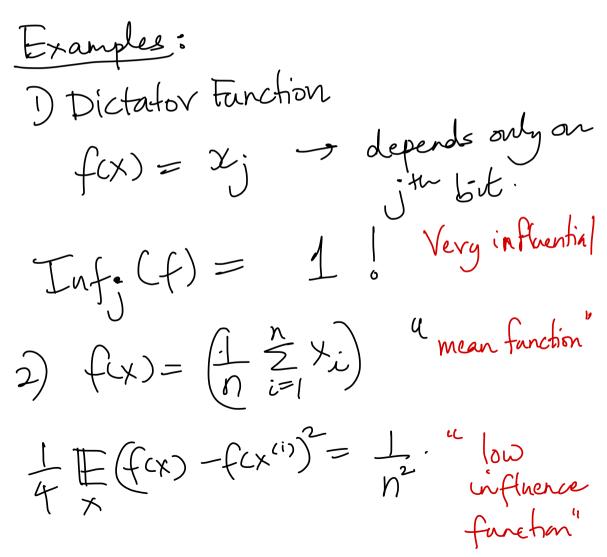




Parseval:



D



f(x) = MAJ(x), n:odd = Sign (\mathcal{Z}_{i}, X_{i}) . When does flipping a bit change the value of MAJ? Ans: when $Z_{i} \times i = 1$ or Zixi=-1. At such an X, any bit flipped will charge the value. Theore the value. Prob $\begin{bmatrix} \Sigma_i X_i = 1 \text{ or } -1 \end{bmatrix} = \begin{pmatrix} n & n \\ n \neq 1 \\ 2^n & 2^n \end{pmatrix}$ $\sim O(\frac{1}{\sqrt{n}})$ All n bits have influence ~ Kn low influence 4) PARITY FUNCTION

 $f(x) = \tilde{T} x_i$ flipping any bit at any x changes the output of PARITY FUNCTION. every lost has influence 1. $\operatorname{Inf}_{i}(f) = | \forall i$

LOW DEGREE INFLUENCE

 $\frac{Def}{Inf_i(f)} = \sum_{\substack{j \in i < c}} \hat{f}(s)^2$ $|S| \leq C$ a discounting the effect of higher degree pairity functions."

Obsh: 7 only C bits can be influential now. $\operatorname{Inf}_{i}(X_{S}) = O$ if 1S > C1 otherwise So, if Isis large, the low degree influence is small.

Lemma (Only a small # vavs can be enfluential) Let f: q-1,13 -> E-1,]. Then, for any C>0 $|\{i \mid Inf(f) \geq \epsilon \}|$ X L W

Proof $\sum_{i=1}^{n} \operatorname{Inf}_{i}^{C} \cdot (f)$ $= 2 2 f(s)^{2}$ $i=1 s \exists i, |s| = c //(1)$ $\leq C \leq f(S) \cdot \leq C$ Since $\leq \hat{f}(s)^2 = \mathbb{E}f(x)^2$

So, by Markov's inequality, frac of i: Inf, Cf) 7, 2 M C N