Hardness of approximation

Recall: undergrad algorithms/complexity

Classes:

polynomial time

$$3SAT \le_m^p \text{VERTEX COVER}$$

mapping reduction.

Def (Mapping Reduction)

$A \le_m^p B$ if there's a poly time algo R such that

$$\begin{cases} \forall\, x \in A, & R(x) \in B \\ \forall\, x \notin A, & R(x) \notin B \end{cases}$$

$$x \in A \Leftrightarrow R(x) \in B.$$

For $A = 3SAT$

$B = k\text{-INDEPENDENT SET}$

If ϕ is satisfiable formula,
then $R(\phi)$ has an IS of size $\geq k$

If ϕ is unsatisfiable formula
then $R(\phi)$ has MIS of size $< k$.

<u>Implication</u>: If there's a poly time algo for k-IS, then there's a poly time algo for 3-SAT

$P \neq NP \Rightarrow \nexists$ poly time algo for 3SAT

$\Rightarrow \nexists$ a poly time algo for k-IS

<u>Key Fact</u>: SAT is NP-Complete
(Cook Levin). How did we prove?
"m/c level proof"

1) SAT served as an amazing starting point for reductions!

2) "Most" problems turned out to be NP-Complete or had a poly time algo ~ so "2" types of problems

HARDNESS OF APPROXIMATION

More granular picture, "many classes" of problems.

Some are constant approx.

$\log(n)$ approx.

n^{ε} approx.

(c,s)-approx curves

bicriteria approx...

<u>Need:</u> 1) a good starting point
2) "approx. preserving" reductions.

Arora-Lund-Motwani-Sudan-Szegedy, Arora-Safra

<u>PCP Theorem '93</u>:

$\exists$ a const $s<1$ s.t. $(1,s)$-3SAT is NP-hard. This $s \sim 10^{-10}$...

$\equiv$ $\forall \phi$ s.t. ϕ is SAT $\rightarrow$ $R(\phi)$ is SAT.

$\forall \phi$ s.t. ϕ is UNSAT $\rightarrow$ no assignment satisfies $> s$ frac of constraints.

Led to a huge # of new hardness of approximation results.

1) Better hardness result? That is smaller s in the hardness result?

2) Is there a poly time algo that can match it?

Raz

1995: Parallel Repetition Theorem

|||

Hardness of Approx. of "Label Cover".

Håstad

1999: Optimal Inapproximability for 3SAT, 3-XOR, k-SAT, ..

Thm: $\forall \varepsilon > 0$, $(1, \frac{7}{8} + \varepsilon)$ - 3SAT is NP-Hard.

Why is this optimal?

Algo: Random Assignment!

Let's start from here and prove hardness of another problem we saw earlier.

We'll first prove a hardness of approx. result for IndependentSet and then use it to derive hardness for Vertex Cover.

<u>Indep set</u>: Given a graph G, set $S \subseteq V$ is independent if there's no edge with both end points in S.

<u>Max Indep set problem</u>: Given G, find the max size I.S. in G

Theorem: There's a poly time reduction algo with that takes input a 3SAT formula Ψ on n vars, m constraints & outputs a graph G on $7m$ vertices and such that:

If Ψ is satisfiable then G has an indep set of size $\geq m$.

If Ψ is $\leq \frac{7}{8}+\varepsilon$ satisfiable, then every indep set of G is of size $\leq \left(\frac{7}{8}+\varepsilon\right)m$.

<u>Proof:</u> "Conflict Graph":

1) Every clause $x_i \vee \bar{x}_j \vee x_k$ has 7 satisfying assignments.

2) For every Clause C and all possible 7 sat assignments α, create a vertex (C, α).

3) Connect $((C,\alpha),(C,\alpha'))$ $\forall \alpha \neq \alpha'$!

4) Connect $((C,\alpha),(C',\alpha'))$ if α & α' conflict.

Claim (Completeness):

Suppose ψ has a sat assignment x.
Then there's an indep set in G of size $\geq m$.

Proof: For each C, choose vertex (C, x_C) to be in S.
Here x_C = assignment induced by x on vars in C.
Then, x_C does not conflict with $x_{C'}$ $\forall C'$.
Thus S is an indep set of size m.

Claim (Soundness) In contrapositive

If S is an indep set in G of size $\geq s \cdot m$, then there is an assignment that satisfies s frac of constraints in Ψ.

Pf: First for any c, there can be at most one (c, α) in S.

Let $c_1, \ldots, c_{sm}$ be the constraints such that $(c_i, \alpha_i) \in S$.

Then, α_i must be all non-conflicting and thus agree with a single assignment x s.t. $x_{c_i} = \alpha_i \; \forall i$. Then x satisfies all c_i.

The two claims above show that $7/8 + \varepsilon$ approx. MIS is NP-hard.

MIN VERTEX COVER

<u>Input</u>: $G(V,E)$ graph, n verts
m edges

<u>Goal</u>: Find <u>smallest</u> $S \subseteq V$ s.t.
S covers all edges. That is
$\forall \{i,j\} \in E, \quad S \cap \{i,j\} \neq \emptyset$.

<u>Obs^n</u>: If $S \subseteq V$ is a V.C. in G then $\overline{S}$ is an I.S. in G.

Thus, $\min VC = n - \max IS$.

Corollary: $\forall \varepsilon > 0$, $1.02-\varepsilon$ approx. V.C. is NP-hard.

Proof: Same reduction from 3SAT as for IS. $\Psi \to G$

We proved: If Ψ has a SAT assignment, then G has an IS of size $\geq m$.

$\Rightarrow \quad V.C.(G) \leq 7m - m = 6m.$

If no assignment satisfies $\geq \frac{7}{8}+\varepsilon$ frac constraints of Ψ, then, every indep set in G has size $\leq \left(\frac{7}{8}+\varepsilon\right)m$

$\Rightarrow \quad V.C.(G) \geq 6m + \frac{1}{8}m - \varepsilon m$

Ratio:
$$\frac{6m + \frac{1}{8}m - \varepsilon m}{6m}$$
$$\sim \frac{49}{48} - \frac{\varepsilon}{6} = 1.02 - \frac{\varepsilon}{6}.$$

Can use other Constraint satisfaction problems instead of 3SAT to improve to 1.16.

DINUR-SAFRA
Thm [2004] 1.34 approx. VC is NP-hard.

(requires some heavy lifting).

DINUR-KHOT-KINDLER-MINZER-SAFRA ...... KHOT-MINZER-SAFRA

Thm [2018]: $\sim\sqrt{2}$-approx. VC is NP hard.

(via proof of 2-to-2 Games Conjecture).

MAX-CUT

Thm 1999: 0.93 approx. Max-Cut is NP-Hard.

Reduction from 3XOR. $\phi(S) = \frac{cap(S,\bar{S})}{dim(S,\bar{S})}$

"Hypercube Gadget"

HARD even on bounded degree graphs.

Alimonty, Kann

Thm [2000]: $(1-\varepsilon)$-approx. Max-Cut on graphs with all degrees ≤ 3 is NP-Hard.

SPARSEST CUT (NON-UNIFORM)

<u>Thm</u>: there is a constant $c > 1$ such that c-approx. (non-uniform) Sparsest cut is NP Hard.

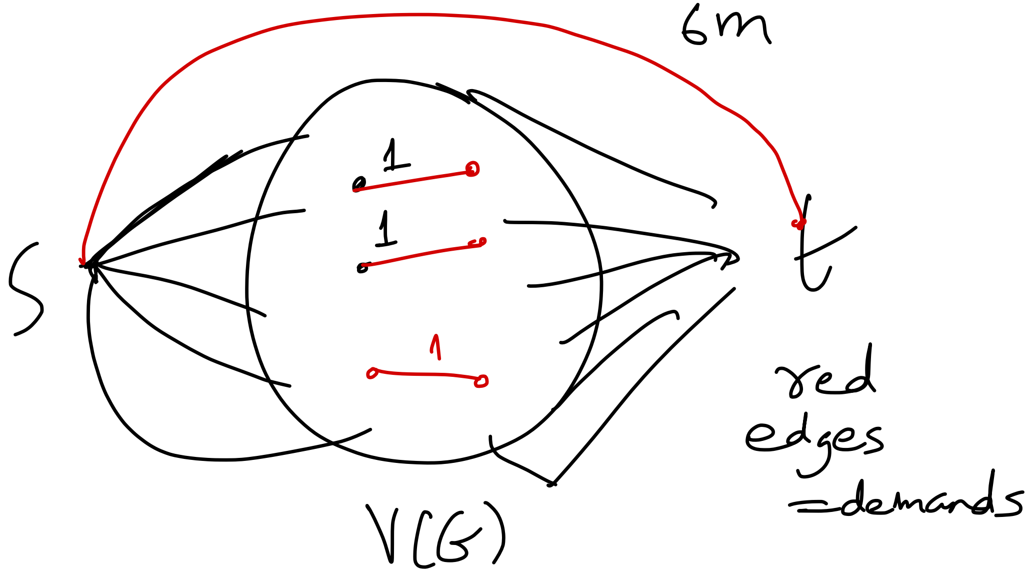
<u>Pf</u>: Reduction from Max-Cut.

G: Instance for Max-Cut.

H: Instance for Sparsest Cut

$$V_H = V_G \cup \underbrace{\{s\} \cup \{t\}}_{\text{"special source \& sink"}}$$

$$E_H = \{s, v\} \ \forall \ v \in V_G$$
$$\{t, v\} \ \forall \ v \in V_G$$



demand pairs: $\forall \{u, v\} \in E_G$, demand of 1 across $\{u, v\}$.

demand of $6m$ across $\{s,t\}$

<u>Claim (Completeness)</u>

Suppose $\text{max-cut}(G) \geq \gamma$

Then, there's a cut in H of sparsity $\alpha = \dfrac{n}{6m + \gamma m}$.

<u>Proof:</u> $S \subseteq V$: a optimal max-cut in G.

then $|E_G(S,\bar{S})| \geq \gamma m$.

Let $S' = \{s\} \cup S$.

Consider cut defined by S' in H.

Then, $|E_H(S', \bar{S}')| = n$. since for every $v \in V(G)$, exactly one of $\{s, v\}$ or $\{t, v\}$ crosses S'.

wt. of demand pairs in the cut $= \gamma m + 6m$

So Sparsity $\alpha = \dfrac{n}{6m + \gamma m}$.

<u>Claim (Soundness)</u>

Suppose that max-cut $(G) \leq (1-\varepsilon)\gamma$ for some $c > 1$.

Then every cut in H has Sparsity $\geq (1 + \Omega(\varepsilon)) \cdot \alpha$

Pf: Say S' is sparsest cut, & $|S'| = k$

Suppose first that s, t are separated by S'. Say WLOG $s \in S'$.

Then, $|E_H(S', \bar{S}')| = n$.

OTOH: wt. of demand pairs in cut

$$\leq (1-\varepsilon) \cdot rm + 6m$$

So. Sparsity of cut $\geq \dfrac{n}{6m + (1-\varepsilon) rm}$

$$= \frac{n}{(1-\varepsilon)(rm + 6m) + \varepsilon \cdot 6 \cdot m} \quad (1)$$

$$\geq \left(1 + \frac{\varepsilon}{18}\right) \cdot \alpha.$$

Now suppose, s & t are ~~both~~ not in S':

Then $|E_H(S',\bar{S'})| = 2\cdot|S'|$

demands met $\leq 3\cdot|S'|$.

all degrees ≤ 3.

So, Sparsity of the cut $\geq \frac{2}{3}$.

Note: $\alpha = \frac{n}{6m+\gamma m} \leq \frac{1}{3(1+\frac{\gamma}{2})} \leq \frac{4}{15}$

$\gamma \geq \frac{1}{2}$ & $m \geq \frac{n}{2}$

$$\frac{2}{3} \geq (2\cdot 5)\cdot\left(\frac{4}{15}\right) \geq \underline{\left(1+\frac{\varepsilon}{18}\right)\cdot\alpha}$$

$\square$.

We saw a bunch of problems &
here's what we know:

	Algo	Hardness
Max-Cut	$\alpha_{GW} = 0.878\ldots$	0.93
Vertex Cover	2	1.41... [2018]
Sparsest Cut (non-uniform)	$O(\sqrt{\log n} \cdot \log\log n)$ ARV	$1+\varepsilon$
uniform	$O(\sqrt{\log n})$	??

For some problems, we had optimal inapproximability: 3SAT, 3XOR, ...

For those on the list above: not so much.