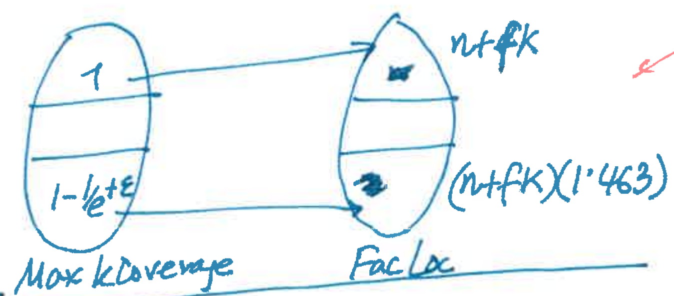
Hardness of Facility location

Know that 1-vs-$(1-1/e+\varepsilon)$ problem for Max Coverage is NP-hard

(i.e. distinguishing instances where k sets cover all of U
vs. k sets cover at most $(1-1/e+\varepsilon)$ of U
is NP hard).

Will use this to show that blah-vs-blah(1.463) problem for Facility Location is hard.

i.e. we will give a map of the form



Shows this apprx is hard as well.

Take instance $(U, \mathcal{S}=\{S_1, S_2 \dots S_m\})$ s.t. $\bigcup S_i = U$.
say $|U| = n$

Construct the fac. loc instance

L	R
1 •	•
2 •	•
•	•
	•
n •	•
elements = clients	sets = "potential facility locations"

- facility costs: $\begin{cases} f_i = \infty & \text{for } i \in L \\ f_i = f & \text{for } i \in R \end{cases}$
- clients $= L$.
- distances: if element $j \in$ set $S_i \Rightarrow d(i,j) = 1$
  else $d(i,j) = 3$.

distance(client, client) = 2
= distance(set, set).

YES instance where k sets cover all elements.

$\Rightarrow$ k facilities cover clients at dist 1.

$$\text{Cost} = k.f + n.1$$

No instance: any αk sets cover at most $n(1-1/e^{\alpha})$ elements.

$\Rightarrow$ in this case cost $\geq f(\alpha k) + n(1 - 1/e^{\alpha}) \cdot 1 + n/e^{\alpha} \cdot 3$

$$= f \cdot \alpha k + \left(1 + \frac{2}{e^{\alpha}}\right) n$$

$\Rightarrow$ best solution $= \min_{\alpha} \left\{ f k \alpha + \left(1 + \frac{2}{e^{\alpha}}\right) n \right\}$

Taking derivatives, get that $\alpha^* = \ln\left(\frac{2n}{fk}\right)$ minimizes this quantity.

$\Rightarrow$ regardless of what # facilities chosen,
cost $\geq fk\alpha^* + \left(1 + \frac{2}{e^{\alpha^*}}\right) n$

<u>Now</u>: our hardness becomes

$$(kf + n) \text{ - vs - } \left(kf \ln\left(\frac{2n}{fk}\right) + n + kf\right)$$

Again, suppose define $x = \frac{kf}{n}$, this becomes

$$n(1+x) \text{ - vs - } n\left(x \ln(2/x) + 1 + x\right)$$

Can set f to maximize this ratio $\dfrac{x \ln(2/x) + 1 + x}{1+x} = 1 + \dfrac{x \ln(2/x)}{1+x}$.

Wolfram Alpha says this is 1.463.

This is the current best hardness result for Facility Location [Guha Khuller '98].

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