Lecture 13: Linear Programming

- Reminder: def.
- Some examples ← max flow
  - shortest paths
  - L_1 regression
- Duality of LPs.

$$\begin{array}{ll} \text{maximize} & \boxed{2x_1 + 3x_2} \\ \text{subject to} & 4x_1 + 8x_2 \le 12 \\ & 2x_1 + x_2 \le 3 \\ & 3x_1 + 2x_2 \le 4 \\ & x_1, x_2 \ge 0. \end{array}$$

objective fn

$l_i(x) \le b_i$

$$\boxed{a_1x_1 + a_2x_2 + \cdots a_nx_n \le b}$$

LP = linear program

"general form"

$$\begin{array}{llrcrcl} \text{maximize} & & 2x_1 & + & 3x_2 & & \\ \text{subject to} & & 4x_1 & + & 8x_2 & \le & 12 \\ & & 2x_1 & + & x_2 & \le & 3 \\ & & 3x_1 & + & 2x_2 & \le & 4 \\ & & x_1, x_2 \ge 0. & & & & \end{array}$$

$$\boxed{\begin{array}{l} \max \ c^T x \\ \quad Ax \le b \\ \quad x \ge 0 \end{array}}$$

componentwise

$$\begin{array}{llrcrcrcl} \text{Minimize} & & 12y_1 & + & 3y_2 & + & 4y_3 & & \\ \text{subject to} & & 4y_1 & + & 2y_2 & + & 3y_3 & \ge & 2 \\ & & 8y_1 & + & y_2 & + & 2y_3 & \ge & 3 \\ & & y_1, y_2, y_3 \ge 0. & & & & & & \end{array}$$

$$\boxed{c^T = \begin{bmatrix} 2 & 3 \end{bmatrix}} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad d^T = \begin{bmatrix} 12 & 3 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 8 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 12 \\ 3 \\ 4 \end{bmatrix} \qquad E = \begin{pmatrix} 4 & 2 & 3 \\ 8 & 1 & 2 \end{pmatrix} \qquad f = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{array}{l} \min c^T x \\ \quad Ax \le b \\ \quad Cx \ge d \\ \quad\quad x \ge 0 \end{array} \qquad \begin{array}{l} \min \ d^T y \\ \text{st } Ey \ge f \\ \quad\quad y \ge 0 \end{array} \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

① max flow as LP $\qquad G=(V,E)$

$$\max \sum_{u:\, su\in E} f(su)$$ ← total flow leaving s

$$\sum_{u:\, uv\in E} f(uv) = \sum_{w:\, vw\in E} f(vw) \quad \forall \text{ vertex } v \neq s,t$$ ← flow conservation

$$0 \le f(e) \le u_e \quad \forall e$$ ← capacities are respected

$$f(e) = 0 \quad \forall \text{ edges entering } s \text{, leaving } t$$

~~$f(e) \in \mathbb{Z}$ $\forall$ $u_e \in \mathbb{Z}$ e~~

Min Cost Max flow:

① solve LP to find max flow value F^*

② $$\min \sum_e c_e f(e)$$

$$\sum_{u: su \in E} f(su) = F^*$$

$$\sum_{u: uv \in E} f(uv) = \sum_{w: vw \in E} f(vw) \qquad \forall v \neq s,t$$

$$0 \leq f(e) \leq u_e \qquad \forall e$$

no flow leaves t, enters s

① figure out vars

② figure out constraints

③ write them down as linear constraints.

L_1-regression

$\boxed{\text{vars: } m, c}$

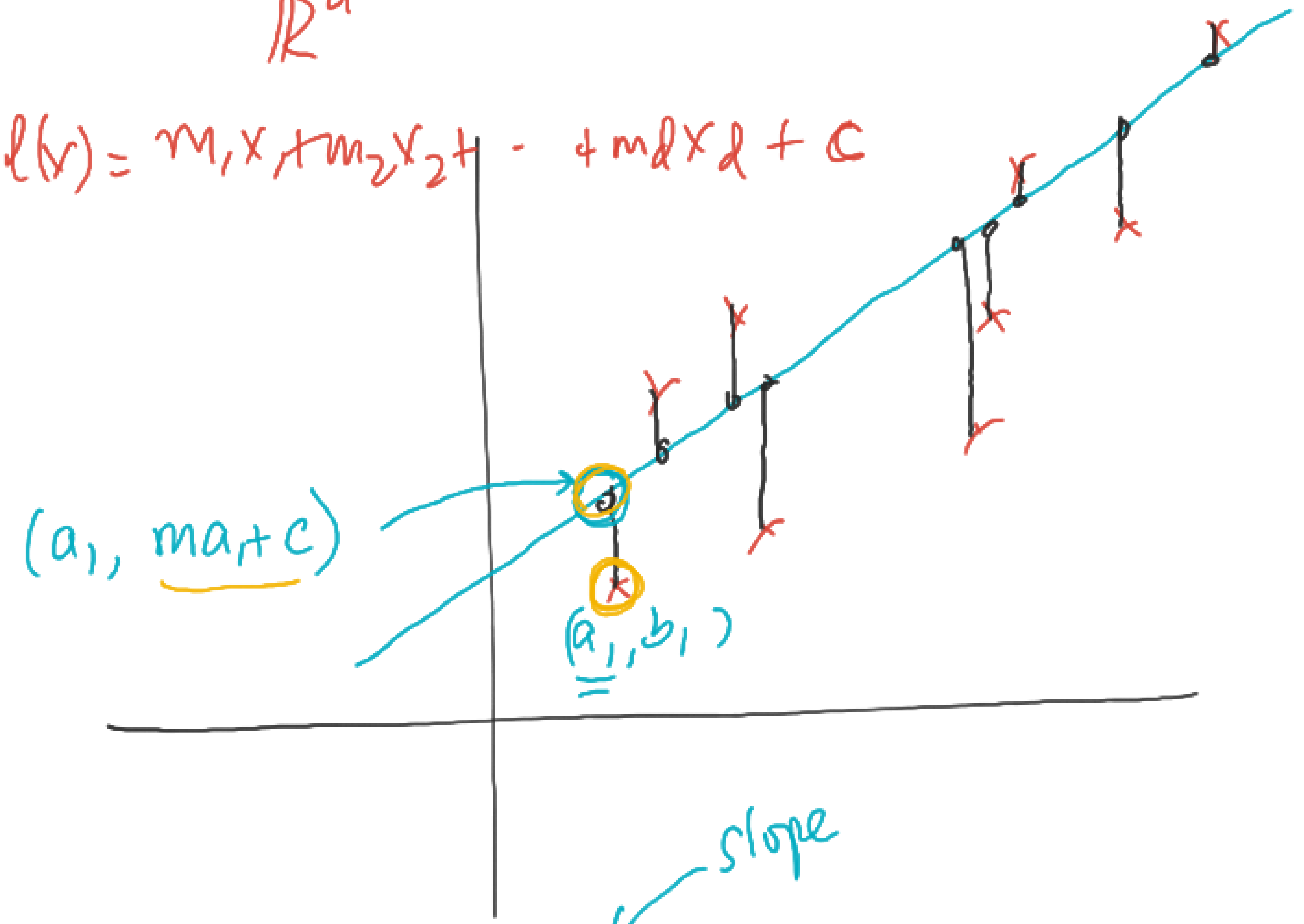
$$\min \sum_{i=1}^{n} \underbrace{\left| b_i - (ma_i + c) \right|^2}$$

absolute value!! not an LP.

data $(a_1, b_1) \cdots (a_n, b_n)$

$\mathbb{R}^d$

$\ell(x) = m_1 x_1 + m_2 x_2 + \cdots + m_d x_d + c$



line = $mx + c$

slope

offset

$$\min \sum_{i=1}^{n} z_i$$

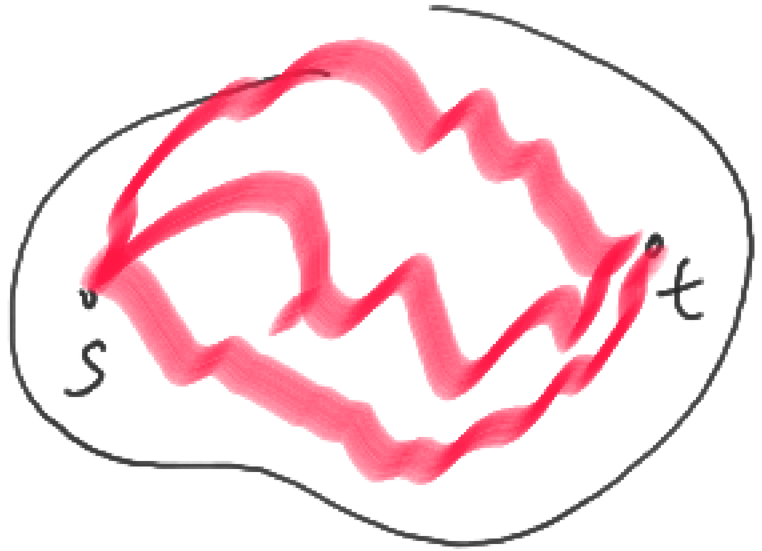
$$\begin{cases} z_i \geq b_i - (ma_i + c) \\ z_i \geq -(b_i - (ma_i + c)) \end{cases}$$

$z_i \geq |b_i - (ma_i + c)|$

Shortest Paths:

directed G, source s, sink t (destination) find shortest s-t path

lengths $l(e) \geq 0$ on edges.



$$\boxed{\begin{aligned} &\min \sum l(e)\, x_e \\ &\sum_{e \text{ leaving } s} x(e) = 1 \\ &\sum_{e \text{ entering } t} x(e) = 1 \\ &\sum_{e \text{ entering } v} x(e) = \sum_{e \text{ leaving } v} x(e) \\ &0 \leq x_e \leq 1 \end{aligned}}$$

solve this min cost flow problem

Every flow carrying path is a shortest path.

variables: distance of vertex v from s. d_v

if want true distances to all nodes from s,

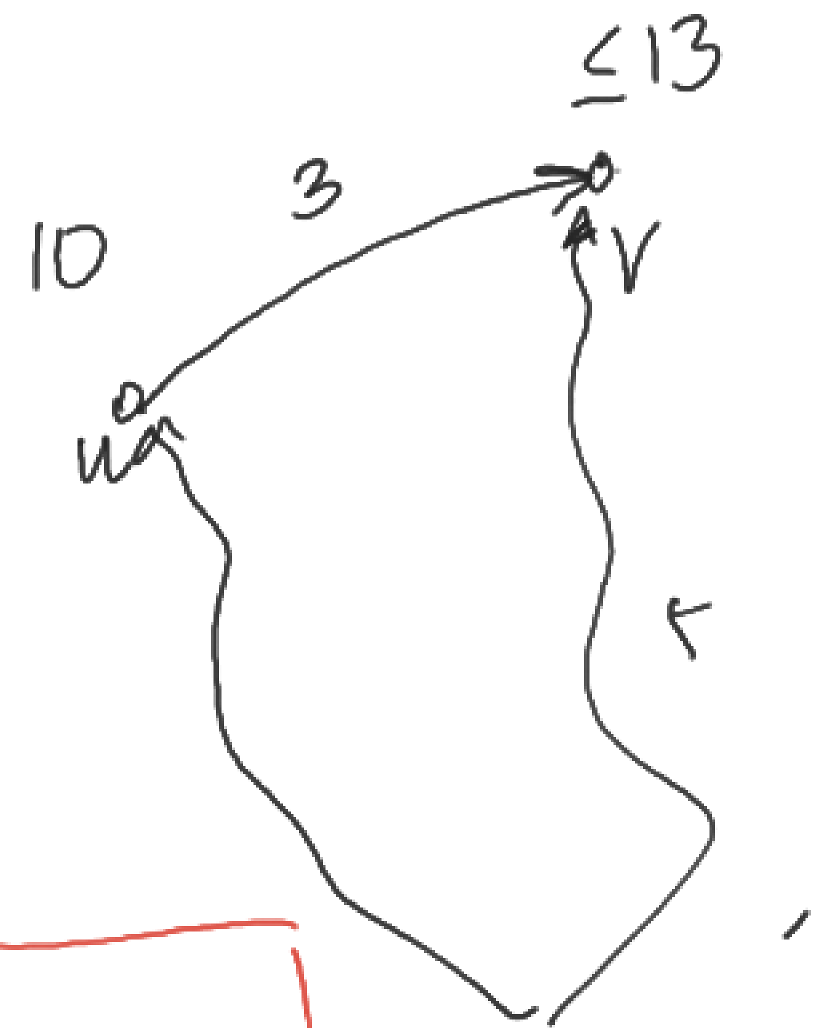
max $\sum_{v \in V} d_v$

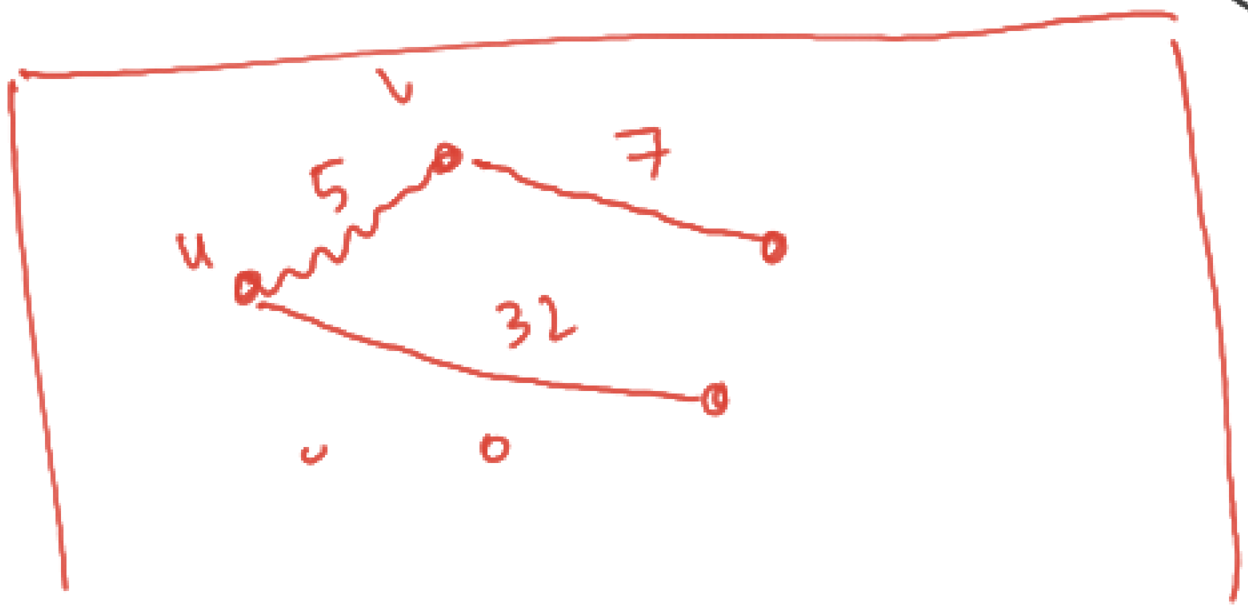
max d_t

$d_v \le d_u + \ell(uv)$ $\forall$ edge $(u,v) \in E$

$d_s = 0$

strings are unstretchable





Duality of LPs

maximize $2x_1 + 3x_2$
subject to
$4x_1 + 8x_2 \le 12$ — y_1
$2x_1 + x_2 \le 3$ — y_2
$3x_1 + 2x_2 \le 4$ — y_3
$x_1, x_2 \ge 0.$

$4y_1 + 2y_2 + 3y_3 \geqslant 2$ $\quad y_1, y_2, y_3 \geq 0$

$8y_1 + y_2 + 2y_3 \geqslant 3$

$\min\ 12y_1 + 3y_2 + 4y_3$

Give me upper bounds on OPT for this LP ?

by nonneg

- $\underline{2x_1 + 3x_2} \le \frac{1}{2}(\underline{4x_1 + 8x_2}) \le \frac{1}{2}12 = 6$

- $\le \frac{1}{3}(c_1 + c_2) \le \frac{1}{3}(12+3) = 5$

LP value

12

7

6

5

OPT = 4.75

$\begin{bmatrix} 2 & 3 \end{bmatrix}$ $\quad \begin{bmatrix} 4 & 8 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$ $\quad \begin{bmatrix} 12 \\ 3 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 12 & 3 & 4 \end{bmatrix}$ $\quad \begin{bmatrix} 4 & 2 & 3 \\ 8 & 1 & 2 \end{bmatrix}$ $\quad \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

*Matoušek Gärtner book

Strong Duality Theorem

Thm: Consider any LP

(primal)

$$\boxed{\begin{array}{ll} \max & c^T x \\ \text{st} & Ax \le b \\ & x \ge 0 \end{array}}$$

then $\exists$ LP (dual)

$$\boxed{\begin{array}{l} \min \ b^T y \\ A^T y \ge c \\ y \ge 0 \end{array}}$$

and they are equal!

primal \ dual	finite	unbounded	infeasible
finite	✓	×	×
unbdd	×	×	✓
infeasible	×	✓	✓

s.t.

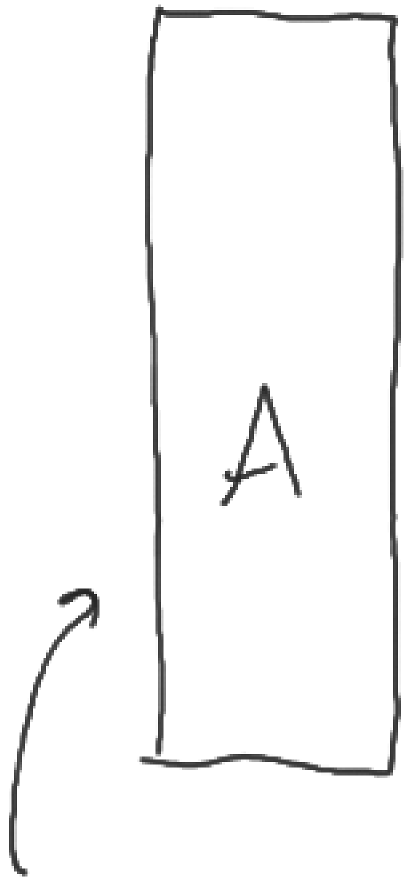
$\max x$	$\min -5y_1 + 3y_2$
$-x \le -5$ $\quad y_1$	st $-y_1 + y_2 \ge 1$
$x \ge 0$ $\quad x \le 3$ $\quad y_2$	$y_1, y_2 \ge 0$

$$\min -5y_1 + 3y_2$$
$$\text{st} \quad -y_1 + y_2 \ge 1$$
$$y_1, y_2 \ge 0$$

max $c^T x$

$Ax \leq b$

$x \geq 0$



m constraints ← except non-neg

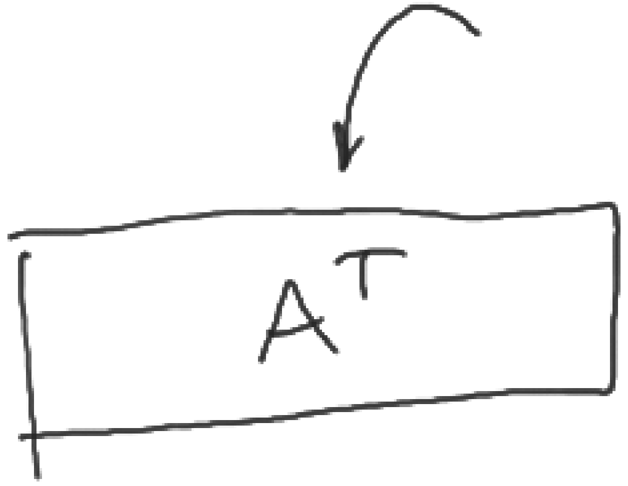
n variables

$A \in \mathbb{R}^{m \times n}$

min $b^T y$

$A^T y \geq c$

$y \geq 0.$



n constraints ← except non-neg

m variable

$$\boxed{\begin{aligned} \max\ & c^T x \\ & Ax \geq b \end{aligned}}$$

$$x = x_1 - x_2$$

$$x_1, x_2 \geq 0$$

def LPs.

$\boxed{\text{modeled problem as LPs}}$

<u>LPs can be solved in poly time!</u>

- simplex
- ellipsoid
- interior point