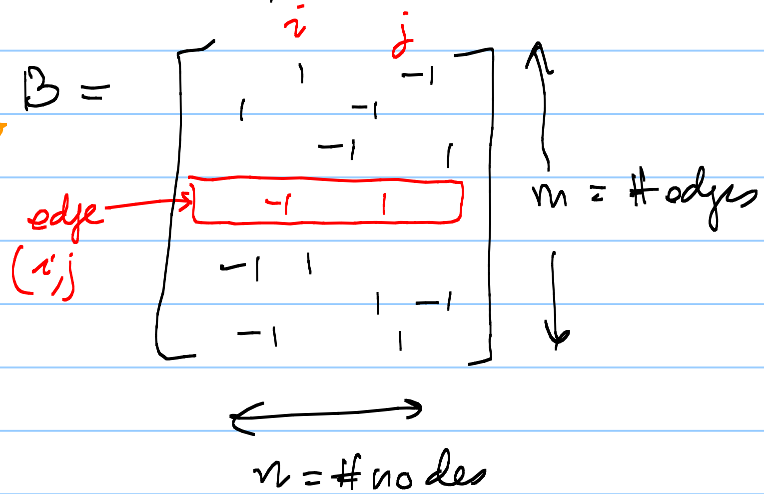
Graphs as Matrices

Note Title 4/23/2020

Building on HW3 problem on Laplacians. —

there we defined $B =$



edge vertex incidence matrix

Each "data point" is an edge with a single $+1$ in col i say
a single -1 in col j say

representing the edge $\{i, j\}$.

[Here we don't worry about the "direction", which is $+1$ and which is -1.]

Today's results will be invariant under this.

Today: consider only undirected → harder to relax graphs.
unweighted → easier to relax

Both can be relaxed

[$G = (V, E)$ n vertices m edges

$B \in \{-1, 0, 1\}^{m \times n}$ edge vertex incidence matrix.

each row has single -1, single $+1$.

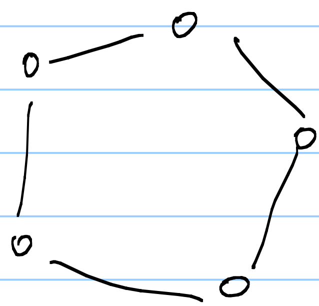
then define

Laplacian $L = B^T B \quad \in \mathbb{R}^{n \times n}$.

Facts: $L = D - A$ ← adjacency matrix $A_{ij} = 1 \iff \{i,j\} \in E$

↑ diagonal matrix with degree of each vertex

E.g. for the cycle C_5, $L(C_5) = \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & 0 & \\ & -1 & 2 & -1 & \\ & 0 & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$



Exercise: L is positive semidefinite $x^T L x \geq 0 \quad \forall x$.

Pf: $x^T L x = x^T B^T B x = \|Bx\|_2^2 \geq 0$.

Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be eigenvalues of L

$v_1 \quad v_2 \quad \quad v_n$ — corresp evectors.

Fact 1: $\lambda_1 = 0$ for $v_1 = \frac{1}{\sqrt{n}} \mathbb{1}$.

Pf: $B\mathbb{1} = 0$ since rowsums are 0.

$\Rightarrow L\mathbb{1} = 0 \cdot \mathbb{1}$

what do the other eigenvalues/vectors tell us?

One more observation

$$\boxed{x^T L x = \sum_{i\sim j}(x_i - x_j)^2 \geq 0.}$$

$i\sim j$ shorthand for $\{i,j\}\in E$.

Very often we consider $\frac{x^T L x}{x^T x} = \frac{x^T L x}{\|x\|^2}$ ← "Rayleigh quotient"

$$= \frac{\sum_{i\sim j}(x_i - x_j)^2}{\sum_{i\in V} x_i^2}$$

——— x ———

A couple examples: K_n has Laplacian $\begin{bmatrix} n-1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & & \vdots \\ & & & -1 & -1 & n-1 \end{bmatrix}$

$$= nI - \mathbb{1}\mathbb{1}^T$$

$\mathbb{1}\mathbb{1}^T$ all 1's matrix

$\lambda_1 = 0$ $\forall$ graphs.

λ_2? $\quad \min_{x\perp\mathbb{1}} \frac{x^T L x}{x^T x}$

but any $x\perp\mathbb{1}$ has $\mathbb{1}^T x = 0 \Rightarrow \mathbb{1}\mathbb{1}^T x = 0$

$$\Rightarrow x^T L x = x^T(nI - \mathbb{1}\mathbb{1}^T)x$$
$$= x^T(nI)x$$
$$= n\|x\|^2$$

$$\Rightarrow \frac{x^T L x}{\|x\|^2} = n. \quad \forall x\perp\mathbb{1}$$

$\Rightarrow$ all other eigenvalues $= n$.

$\sum_i \lambda_i = \sum_i d_i$ egree $= tr(L)$

For complete graph $\lambda_1 = 0$ $\lambda_2 = \lambda_3 \ldots \lambda_n = n$

So for "very connected" graphs, gap between λ_2 and $\lambda_1 = 0$ is large

In fact can show: $\begin{cases} \lambda_2 \leq \frac{n}{n-1} & \rightarrow \lambda_2 \leq \frac{\sum_i d_i}{n-1} \\ \lambda_n \leq 2d_{max}, & \text{and } \lambda_n \leq n \text{ for simple graphs} \end{cases}$ [See Mohar survey]

So this is the largest gap possible.

$\lambda_n \geq \frac{\sum_i d_i}{n-1}$

Next: λ_2 captures connectivity

Fact 2: 0 appears as a repeated eigenvalue $\Leftrightarrow$ G is not connected

in fact multiplicity of 0 as evalue = # connected comps.

Pf: sps k components. $V_1, V_2 \ldots V_k$ be partition of V

then $v_i = \begin{cases} 1\text{s on } i^{th} \text{ component} \\ 0\text{s everywhere else} \end{cases}$ is an Evector for $i \leq k$

and $v_i \perp v_j$ for all these $i \neq j$

So at least dimension k for eigenspace of evalue 0.

Can there be more? No.

Sps. $\exists\, x$ orthogonal to $v_1 v_2 \ldots v_k$. with eval $= 0$.

then x has non zero value on some vertex (say in V_i)

all must be 0

But then since $x^T L x = 0 \iff \sum_{u \sim v} (x_u - x_v)^2 = 0$

$\Rightarrow x_u = x_v \;\; \forall u, v \in V_i$

this means $x \not\perp v_i$ since both are constant on V_i.

$\blacksquare$

So: No gap between $\lambda_1, \lambda_2 \Rightarrow$ graph is not connected

$\lambda_1 \dots \lambda_k \Rightarrow$ graph has k components

Why is this interesting?

Actually: not just qualitatively, but quantitatively important

the spectral gap $\lambda_2 - \lambda_1$ captures the "degree of connectivity" of G.

$\lambda_2 - \lambda_1$ is large $\iff$ G is "well connected"

in the sense of not having small "balanced" cuts.

We'll see details in a moment when we talk about Cheeger.

Do graphs with poor connectivity indeed have small spectral gap $\lambda_2 - \lambda_1$?

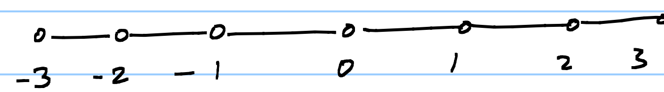
Let's do an example or two.

Path graph

$\lambda_2 = \Theta\left(\frac{1}{n^2}\right)$ and this is within constants of smallest gap for connected simple graphs

Lets show: $\lambda_2 \leq O\left(\frac{1}{n^2}\right)$ the lower bound requires some work

Show a "test" vector $v \perp \mathbb{1}$. (n: say odd)

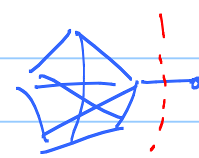


$\Rightarrow$ Rayleigh quotient $= \dfrac{\sum_{ij}'(x_i - x_j)^2}{\sum_i' x_i^2} = \dfrac{n-1}{2(1^2+2^2+\dots+\frac{n}{2}^2)} \cong \dfrac{n}{\Theta(n^3)} \cong \Theta\left(\frac{1}{n^2}\right)$.

$\lambda_2 - \lambda_1 = \lambda_2 - 0 = \lambda_2$

In fact: spectral gap captures the algebraic connectivity.

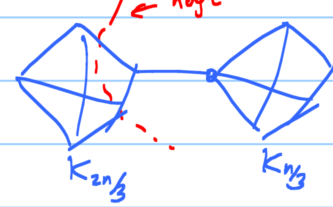
What notion of graph connectivity



Min Cut: cut least # of edges to separate graph into 2.

← huge bisection cost.

Bisection: cut ———— to separate into 2 equal parts

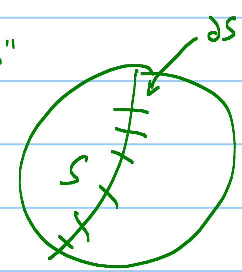


Here: "conductance" of a cut $(S, \bar{S})$ in G

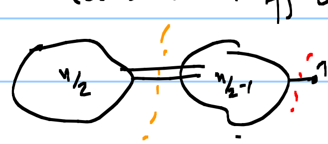
"boundary of S"

$$\phi_G(S) = \frac{\#\text{ edges crossing the cut}}{\#\text{ vertices in } S} = \frac{|\delta_G(S)|}{|S|}$$

← often just say ∂S

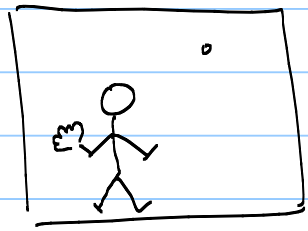


Fact: computing min conductance cuts often good in applications (image segmentation)
(doesnot cut off small inconsequential objects say)



(prefers orange to red cut)

Eg



Will segment out player before ball

$\phi(G)$ = min conductance cut $= \min\limits_{|S|\le n/2} \phi_G(S) = \min\limits_{S\subseteq V} \dfrac{|\partial S|}{\min(|S|,|\bar{S}|)}$

$$\simeq \min_{S\subseteq V} \left(\frac{|\partial S|}{|S| \times (n-|S|)} \cdot n \right)$$

up to a factor of 2 (since $\min |S|,|\bar{S}| \le n/2$
so other guy is $\ge n/2$.

$$\frac{\phi(G)}{n} \simeq \min_{\substack{x\in\{0,1\}^n \\ x\neq \text{all zero}}} \frac{\sum'_{i\sim j}(x_i-x_j)^2}{\sum'_{i,j}(x_i-x_j)^2}$$

(factor of 2)

$$\ge \min_{x \text{ nontrivial}} \frac{\sum'_{i\sim j}(x_i-x_j)^2}{\sum'_{i,j}(x_i-x_j)^2}$$

relaxation!

but since differences don't change if we shift every x by same c, make $x^{new} \leftarrow x-c$ st. $x^{new} \perp \mathbb{1}$.

$$= \min_{x\perp \mathbb{1}} \frac{\sum'_{i\sim j}}{\sum'_{i,j}}$$

$$\left(n\sum_i' x_i^2 + n\sum_j' x_j^2 - 2\sum_{i,j}' x_i x_j\right)$$

$$\left(\sum_i' x_i\right)\left(\sum_j' x_j\right) = 0$$ since $\sum' x_i = \langle x, \mathbb{1}\rangle = 0$.

when the dust settles...

$$\frac{\phi(G)}{n} \simeq \min_{x\perp \mathbb{1}} \frac{\sum_{i\sim j}(x_i-x_j)^2}{2n \cdot \sum_i' x_i^2} = \frac{\lambda_2}{2n}$$

factor of 2 :

Theorem: Let G be a graph with max degree d_{max}

Cheeger's Ineq.

$$\Rightarrow \quad \frac{\phi(G)^2}{d_{max}} \leq \lambda_2 \leq 2\phi(G)$$

can get rid of this using "normalized Laplacians"

the square is inherent.