Lecture 18:
Memory Hierarchy Optimizations & Locality Analysis

[ALSU 7.4.2-7.4.3, 11.2-11.5]

Caches: A Quick Review

- How do they work?
- Why do we care about them?
- What are typical configurations today?
- What are some important cache parameters that will affect performance?

Optimizing Cache Performance

- Things to enhance:
  - temporal locality
  - spatial locality
- Things to minimize:
  - conflicts (i.e. bad replacement decisions)

What can the *compiler* do to help?

Two Things We Can Manipulate

- Time:
  - When is an object accessed?
- Space:
  - Where does an object exist in the address space?

How do we exploit these two levers?

Time: Reordering Computation

- What makes it difficult to know *when* an object is accessed?
- How can we predict a better time to access it?
  - What information is needed?
- How do we know that this would be safe?

Space: Changing Data Layout

- What do we know about an object's location?
  - scalars, structures, pointer-based data structures, arrays, code, etc.
- How can we tell what a better layout would be?
  - how many can we create?
- To what extent can we safely alter the layout?

Types of Objects to Consider

- Scalars
- Structures & Pointers
- Arrays

Scalars

- Locals
- Globals
- Procedure arguments
- Is cache performance a concern here?
- If so, what can be done?

```
int x;
double y;
foo(int a){
  int i;
  …
  x = a*i;
  …
}
```

Structures and Pointers

- What can we do here?
  - within a node
  - across nodes

```
struct {
   int count;
   double velocity;
   double inertia;
   struct node *neighbors[N];
} node;
```

Example: Can rearrange field order to improve cache performance

- What limits the compiler’s ability to optimize here?

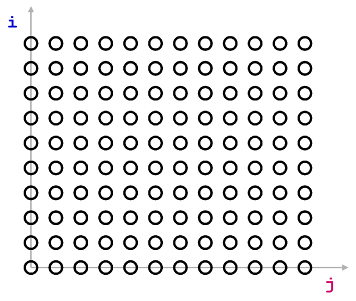
Arrays / Matrices

```
double A[N][N], B[N][N];
…
for i = 0 to N-1
   for j = 0 to N-1
      A[i][j] = B[j][i];
```

- usually accessed within loops nests
  - makes it easy to understand “time”
- what we know about array element addresses:
  - start of array?
  - relative position within array

Handy Representation: “Iteration Space”

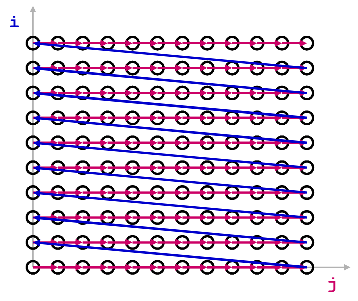
```
for i = 0 to N-1
   for j = 0 to N-1
      A[i][j] = B[j][i];
```



- each position represents an iteration (not an array element)

Visitation Order in Iteration Space

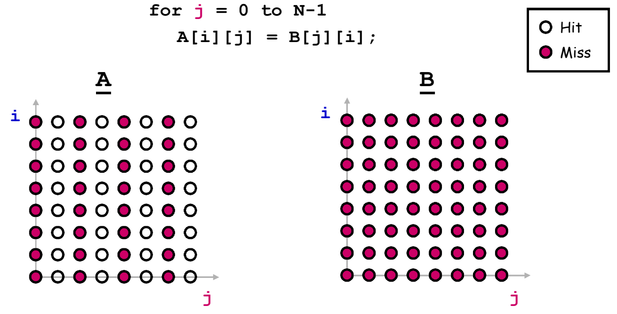
```
for i = 0 to N-1
   for j = 0 to N-1
      A[i][j] = B[j][i];
```



- Note: iteration space ≠ data space

When Do Cache Misses Occur?

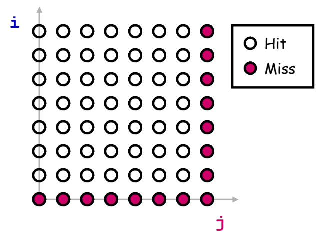
```
for i = 0 to N-1
   for j = 0 to N-1
      A[i][j] = B[j][i];
```



Assume row major order, N large, 2 elements per cache line

When Do Cache Misses Occur?

```
for i = 0 to N-1
   for j = 0 to N-1
      A[i+j][0] = i*j;
```



Row major layout of A:
A[0][0] A[0][1]…A[0][N-1] A[1][0]…A[1][N-1]…A[2N-2][0]…A[2N-2][N-1]

Optimizing the Cache Behavior of Array Accesses

- We need to answer the following questions:
  - when do cache misses occur?
    - use "locality analysis"
  - can we change the order of the iterations (or possibly data layout) to produce better behavior?
    - evaluate the cost of various alternatives
  - does the new ordering/layout still produce correct results?
    - use "dependence analysis"

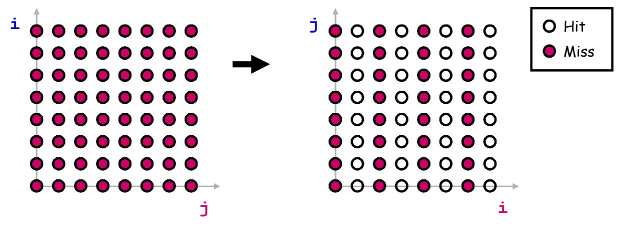
Examples of Loop Transformations

- Loop Interchange
- Cache Blocking
- Skewing: iterate thru iteration space in the loops at an angle
- Loop Reversal: execute iterations in a loop in reverse order
- …

(we will briefly discuss the first two;
see ALSU 11.7.8 for others)

Loop Interchange

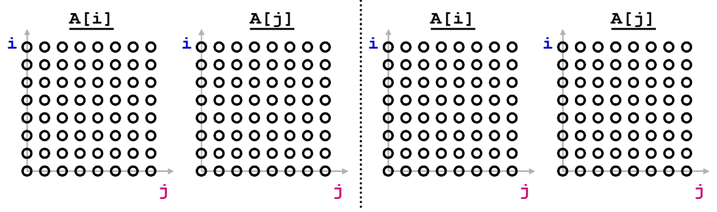
```
for i = 0 to N-1                for j = 0 to N-1
  for j = 0 to N-1                for i = 0 to N-1
    A[j][i] = i*j;                  A[j][i] = i*j;
```



- *(assuming 2 elements/cache line & N is large relative to cache size)*

Cache Blocking (aka "Tiling")

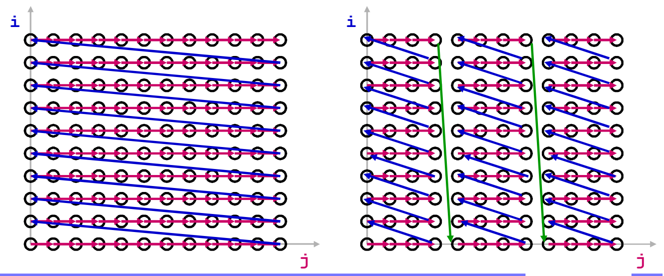
```
                                for JJ = 0 to N-1 by L
for i = 0 to N-1                  for i = 0 to N-1
  for j = 0 to N-1                  for j = JJ to max(N-1,JJ+L-1)
    f(A[i],A[j]);                     f(A[i],A[j]);
```



now we can exploit temporal locality

Impact on Visitation Order in Iteration Space

```
                                for JJ = 0 to N-1 by L
for i = 0 to N-1                  for i = 0 to N-1
  for j = 0 to N-1                  for j = JJ to max(N-1,JJ+L-1)
    f(A[i],A[j]);                     f(A[i],A[j]);
```



Cache Blocking in Two Dimensions

```
                                   for JJ = 0 to N-1 by B
                                     for KK = 0 to N-1 by B
for i = 0 to N-1                       for i = 0 to N-1
  for j = 0 to N-1                       for j = JJ to max(N-1,JJ+B-1)
    for k = 0 to N-1                       for k = KK to max(N-1,KK+B-1)
      c[i,k] += a[i,j]*b[j,k];               c[i,k] += a[i,j]*b[j,k];
```

- brings square sub-blocks of matrix "b" into the cache
- completely uses them up before moving on
- reduces the number of misses from $\frac{N^3}{L}$ or N^3 to only $\frac{2N^3}{L\,C}$ (C=cache size, L=line size)

Predicting Cache Behavior through "Locality Analysis"

- Definitions:
  - Reuse:
    - accessing a location that has been accessed in the past
  - Locality:
    - accessing a location that is now found in the cache
- Key Insights
  - Locality only occurs when there is reuse!
  - BUT, reuse does not necessarily result in locality.
    - why not?

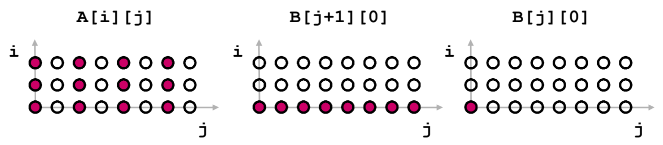
Steps in Locality Analysis

1. Find data reuse
   - if caches were infinitely large, we would be finished
2. Determine "localized iteration space"
   - set of inner loops where the data accessed by an iteration is expected to fit within the cache
3. Find data locality:
   - reuse $\cap$ localized iteration space $\Rightarrow$ locality

Types of Data Reuse/Locality

```
for i = 0 to 2
  for j = 0 to 100
    A[i][j] = B[j][0] + B[j+1][0];
```

○ Hit
● Miss



Spatial Temporal Group

(assume 2 elements per cache line)

Reuse Analysis: Representation

```
for i = 0 to 2
  for j = 0 to 100
    A[i][j] = B[j][0] + B[j+1][0];
```

- Map *n* loop indices into *d* array indices via array indexing function:

$$\vec{f}(\vec{i}) = H\vec{i} + \vec{c}$$

$$\texttt{A[i][j]} = \texttt{A}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

$$\texttt{B[j][0]} = \texttt{B}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

$$\texttt{B[j+1][0]} = \texttt{B}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

Finding Temporal Reuse

- Temporal reuse occurs between iterations $\vec{\imath}_1$ and $\vec{\imath}_2$ whenever:

$$H\vec{\imath}_1 + \vec{c} = H\vec{\imath}_2 + \vec{c}$$
$$H(\vec{\imath}_1 - \vec{\imath}_2) = \vec{0}$$

- Rather than worrying about individual values of $\vec{\imath}_1$ and $\vec{\imath}_2$ we say that reuse occurs along direction vector $\vec{r}$ when:

$$H(\vec{r}) = \vec{0}$$

- Solution: compute the *nullspace* of *H*

Temporal Reuse Example

```
for i = 0 to 2
  for j = 0 to 100
    A[i][j] = B[j][0] + B[j+1][0];
```

- Reuse between iterations (i_1,j_1) and (i_2,j_2) whenever:

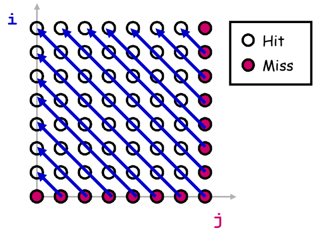
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} i_1 \\ j_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} i_2 \\ j_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} i_1 - i_2 \\ j_1 - j_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- True whenever $j_1 = j_2$, and regardless of the difference between i_1 and i_2.
  - i.e. whenever the difference lies along the nullspace of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which is span{(1,0)} (i.e. the outer loop).

More Complicated Example

```
for i = 0 to N-1
  for j = 0 to N-1
    A[i+j][0] = i*j;
```

$$\texttt{A[i+j][0]} = \texttt{A}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$



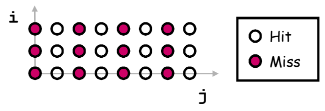
- Nullspace of $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ = span{(1,-1)}, i.e. when $\Delta i = -\Delta j$.

Computing Spatial Reuse

- Assume two array elements share the same cache line iff they differ only in the last dimension
  - E.g., share the same row in a 2-dimensional array
  - Why is this a reasonable approximation? row major order
  - What are its limitations? May jump around too much within row
- Replace last row of *H* with zeros, creating H_s
- Find the nullspace of H_s
- Result: vector along which we access the same row

Computing Spatial Reuse: Example

```
for i = 0 to 2
  for j = 0 to 100
    A[i][j] = B[j][0] + B[j+1][0];
```



$$\texttt{A[i][j]} = \texttt{A}\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}i\\ j\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}\right)$$

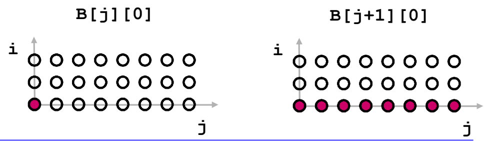
- $H_s = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$
- Nullspace of H_s = span{(0,1)}, i.e., the inner loop
  - access same row of **A[i][j]** along inner loop

Group Reuse (reuse from different static accesses)

```
for i = 0 to 2
  for j = 0 to 100
    A[i][j] = B[j][0] + B[j+1][0];
```

$H = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$

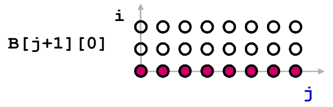
- Limit the analysis to consider only accesses with same H
  - i.e., index expressions that differ only in their constant terms
- Determine when access same location (temporal) or same row (spatial)
- Only the "leading reference" suffers the bulk of the cache misses



Localized Iteration Space

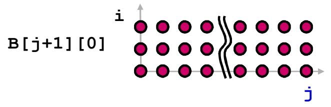
- Given finite cache, when does reuse result in locality?

```
for i = 0 to 2
  for j = 0 to 8
    A[i][j] = B[j][0] + B[j+1][0];
```



Localized: both i and j loops

```
for i = 0 to 2
  for j = 0 to 1000000
    A[i][j] = B[j][0] + B[j+1][0];
```



Localized: j loop only

- Localized if accesses less data than *effective cache size*

Computing Locality

- Reuse Vector Space $\cap$ Localized Vector Space $\Rightarrow$ Locality Vector Space
- Example:

```
for i = 0 to 2
  for j = 0 to 100
    A[i][j] = B[j][0] + B[j+1][0];
```

- If both loops are localized:
  - span{(1,0)} $\cap$ span{(1,0),(0,1)} $\Rightarrow$ span{(1,0)}
  - i.e. temporal reuse does result in temporal locality
- If only the innermost loop is localized:
  - span{(1,0)} $\cap$ span{(0,1)} $\Rightarrow$ span{}
  - i.e. no temporal locality