Algorithm Design and Analysis

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Lecture 11 Feb 07, 2014 Carnegie Mellon University

Graph Algorithms

Plan:

DFS
Topological Sorting
Classification of Edges
Biconnected Components

Graphs Traversal

Visiting all vertices in a systematic order.

```
for all v in V do visited[v] = false
for all v in V do if !visited[v] traversal(v)

  traversal(v)  {                              O(V + E)
    visited[v] = true
    for all w in adj(v)
        do if !visited[w] traversal(w)
  }
```

Graphs Traversals

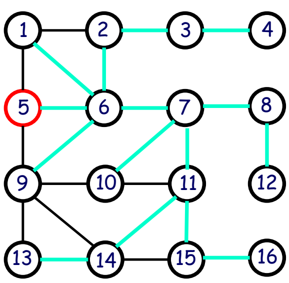
- Depth-First Search (DFS)
- Breadth-First Search (BFS)

DFS uses a stack for backtracking.
BFS uses a queue for bookkeeping

Properties of DFS

Property 1
DFS visits all the vertices in the connected component

Property 2
The discovery edges labeled by DFS form a spanning tree of the connected component

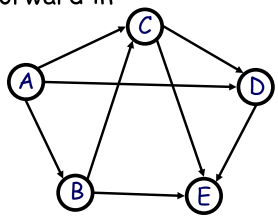


Applications of DFS

- Determine the connected components of a graph
- Find cycles in a graph
- Determine if a graph is bipartite.
- Topologically sort in a directed graph
- Find the biconnected components

Topological Sorting

Find an ordering of the vertices such that all edges go forward in the ordering.



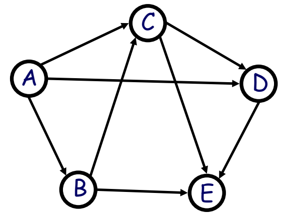
It's easy to see that such an ordering exists.
Find a vertex with zero in-degree. Print it, delete it from the graph, and repeat.

Complexity-?

PQ wrt in-degrees. O(E log V)

Topological Sorting with DFS

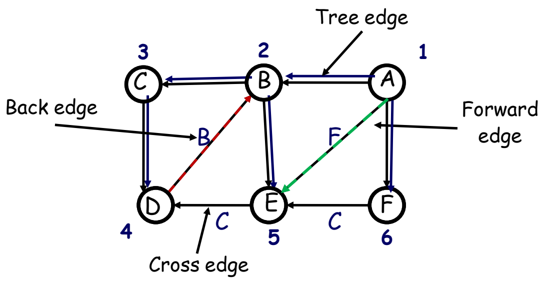
```
DFS (v) {
  visited[v] = true
  for all w in adj(v)
    do if !visited[w]
          DFS (w);
  print(v);
  }
Do DFS;
Reverse the order;
```



Complexity-?

O(E + V)

Classification of Edges with DFS



Classification of Edges

Tree edges - are edges in the DFS

Forward edges – edges (u,v) connecting u to a descendant v in a depth-first tree

Back edges – edges (u,v) connecting u to an ancestor v in a depth-first tree

Cross edges – all other edges

DAG

Theorem.
A directed graph is acyclic iff a DFS yields no back edges.

DAG

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A directed graph is acyclic iff a DFS yields no back edges.

Proof.
=>) by contrapositive.
If there is a back edge, the graph is surely cyclic.

Theorem.

A directed graph is acyclic iff a DFS yields no back edges.

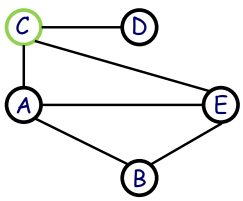
Proof.

<=) Suppose there is a cycle.

Let v be the first vertex discovered in the cycle. Let (u, v) be the preceding edge in this cycle. When we push v on the stack, no any vertices on the cycle were discovered yet. Thus, vertex u becomes a descendent of v in DFS. Therefore, (u, v) is a back edge.

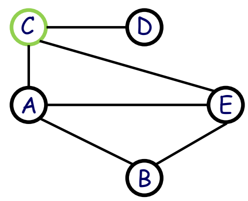
```
for all v in V do num[v] = 0, stack[v]=false
for all v in V do if num[v]==0 DFS(v)
k = 0;

  DFS(v)  {
    k++; num[v] = k; stack[v]=true
    for all w in adj(v)  do
       if num[w]==0 DFS(w)          tree edge
       else if num[w]  > num[v]     forward edge
       else if stack[w]             back edge
       else                         cross edge
    stack [v]=false
  }
```

Biconnectivity



In many applications it's not enough to know that a graph is connected, but "how well" it's connected.

Articulation points

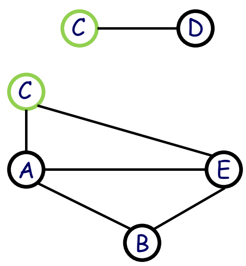


A vertex is an articulation point if its removal (with edges) disconnect a graph.

A connected graph is biconnected if it has no articulation points.

If a graph is not biconnected, we define the biconnected components

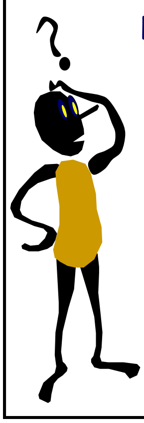
Biconnected Components



If a graph is not biconnected, we define the biconnected components

Biconnected graphs are of great interest in communication and transportation networks

Find articulation points



Fred Hacker's algorithm:

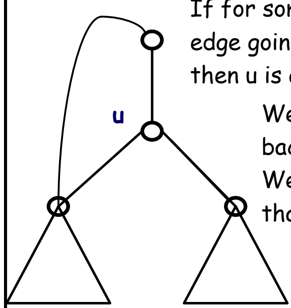
Delete a vertex
Run DFS to see if a graph is connected
Choose a new vertex. Repeat.

Complexity: O(V (V+E))

Biconnected Component Algorithm

- It is based on a DFS
- We assume that G is undirected and connected.
- We cannot distinguish between forward and back edges
- Also there are no cross edges (!)

Find articulation point: an observation



If for some child, there is no back edge going to an ancestor of u, then u is an articulation point.

We need to keep a track of back edges!
We keep a track of back edge that goes higher in the tree.

Find articulation point: next observation

What about the root?
Can it be an articulation point?

DFS root must have two or more children

Biconnected Component Algorithm

- Run DFS
- When we reach a dead end, we will back up. On the way up, we will discover back edges. They will tell us how far in the tree we could have gone.
- These back edges indicate a cycle in the graph. All nodes in a cycle must be in the same component.

Bookkeeping

- For each vertex we will store two indexes. One is the counter of nodes we have visited so far dfs[v]. Second - the back index low[v].
- Definition.

low[v] is the DFS number of the lowest numbered vertex x (i.e. highest in the tree) such that there is a back edge from some descendent of v to x.

How to compute low[v]?

- Tree edge (u, v)

  low[u] = min(low[u], low[v])

Vertices u and v are in the same cycle.

- Back edge (u, v)

  low[u] = min(low[u], dfs[v])

If the edge goes to a lower dfs value then the previous back edge, make this the new low.

How to test for articulation point?

Using low[u] value we can test whether u is an articulation point.

If for some child, there is no back edge going to an ancestor of u, then u is an articulation point.

If there was a back edge from child v, than low[v] < dfs[u].

It follows, u is an articulation point iff it has a child v such that low[v] >= dfs[u].

The Algorithm

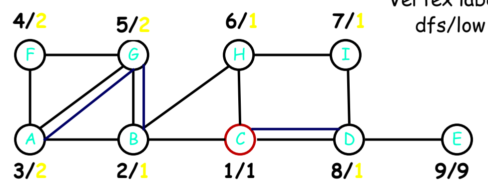
low(A) = dfs(B)

Remove bicomponent GFAB

All edges are on a stack

Store edges on a stack as you run DFS

Vertex labels dfs/low



Theorem : Let G = (V, E) be a connected, undirected graph and S be a depth-first tree of G. Vertex x is an articulation point of G if and only if one of the following is true:

(1) x is the root of S and x has two or more children in S.

(2) x is not the root and for some child s of x, there is no back edge between any descendant of s (including s itself) and a proper ancestor of x.

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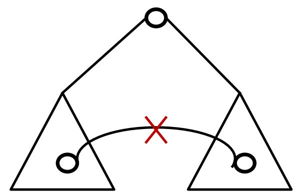
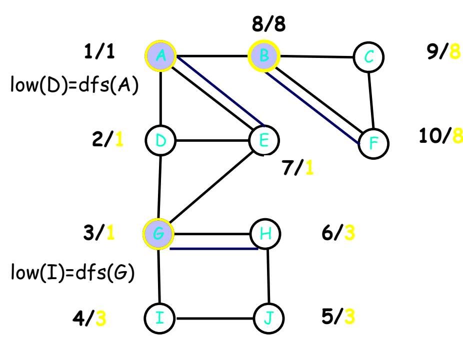


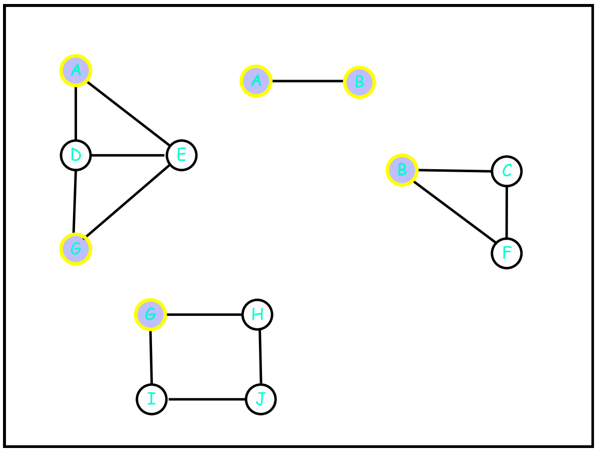
Theorem : Let G = (V, E) be a connected, undirected graph and S be a depth-first tree of G. Vertex x is an articulation point of G if and only if one of the following is true:

(2) x is not the root and for some child s of x, there is no back edge between any descendant of s (including s itself) and a proper ancestor of x.

Proof: =>) If x is an articulation vertex, then removing it will disconnect child s from the parent of x.

<=) If there is no such s, then x is not articulation point. To see this, suppose v_0 is the parent and $v_1, \ldots, v_k$ are all children. By our assumption, there exists a path from v_i to v_0. They are in the same connected components. Removing x, won't disconnect the graph.

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