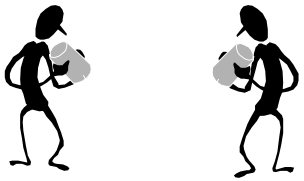
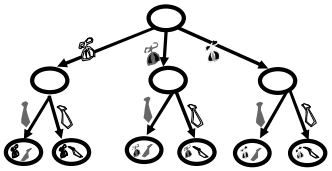
| Great Theoretical Ideas In Computer Science | | |
|---|---|---|
| Steven Rudich, Anupam Gupta | | CS 15-251 Spring 2005 |
| Lecture 22 | March 31, 2005 | Carnegie Mellon University |

# Decision Trees and Information: A Question of Bits

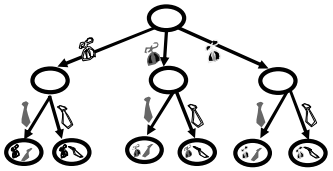


## Choice Tree



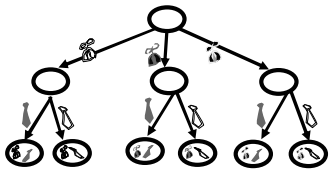
A choice tree is a rooted, directed tree with an object called a "choice" associated with each edge and a label on each leaf.

## Choice Tree Representation of S



We satisfy these two conditions:
- Each leaf label is in S
- Each element from S on exactly one leaf.

## Question Tree Representation of S



I am thinking of an outfit.
Ask me questions until you know which one.

What color is the beanie?
What color is the tie?

When a question tree has at most 2 choices at each node, we will call it a decision tree, or a decision strategy.

Note: Nodes with one choices represent stupid questions, but we do allow stupid questions.

## 20 Questions

S = set of all English nouns

Game:
I am thinking of an element of S.
You may ask up to 20 YES/NO questions.

What is a question strategy for this game?

## 20 Questions

Suppose $S = \{a_0, a_1, a_2, \ldots, a_k\}$

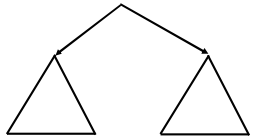
Binary search on S.

First question will be:

"Is the word in $\{a_0, a_1, a_2, \ldots, a_{(k-1)/2}\}$ ?"

## 20 Questions
## Decision Tree Representation

A decision tree with depth at most 20, which has the elements of S on the leaves.



Decision tree for $\{a_0, a_1, a_2, \ldots, a_{(k-1)/2}\}$

Decision tree for $\{a_{(k+1)/2}, \ldots, a_{k-1}, a_k\}$

## Decision Tree Representation

Theorem:
The binary-search decision tree for S with k+1 elements $\{ a_0, a_1, a_2, \ldots, a_k \}$ has depth

$$\lceil \log (k+1) \rceil$$
$$= \lfloor \log k \rfloor + 1$$
$$= |k|$$

"the length of k when written in binary"

## Another way to look at it

Suppose you are thinking of the noun $a_m$ in S
We ask about each bit of index m

Is the leftmost bit of m 0?
Is the next bit of m 0?
...

Theorem: The binary-search decision-tree for $S = \{ a_0, a_1, a_2, \ldots, a_k \}$ has depth

$$|k| = \lfloor \log k \rfloor + 1$$

## A lower bound

Theorem: No decision tree for S (with k+1 elements) can have depth $d < \lfloor \log k \rfloor + 1$ .

Proof:
A depth d binary tree can have at most $2^d$ leaves.
But $d < \lfloor \log k \rfloor + 1 \Rightarrow$ number of leaves $2^d < (k+1)$
Hence some element of S is not a leaf.

## Tight bounds!

The optimal-depth decision tree
for any set S with (k+1) elements has depth

$$\lfloor \log k \rfloor + 1 = |k|$$

## Recall…

The minimum number of bits used to represent unordered 5 card poker hands =

$$\lceil \log_2 \binom{52}{5} \rceil$$

= 22 bits

= The decision tree depth for 5 card poker hands.

## Prefix-free Set

Let T be a subset of {0,1}*.

Definition:
T is prefix-free if for any distinct x,y ∈ T,
if |x| < |y|, then x is not a prefix of y

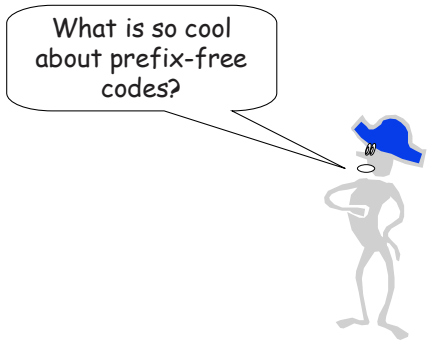
Example:
{000, 001, 1, 01} is prefix-free
{0, 01, 10, 11, 101} is not.

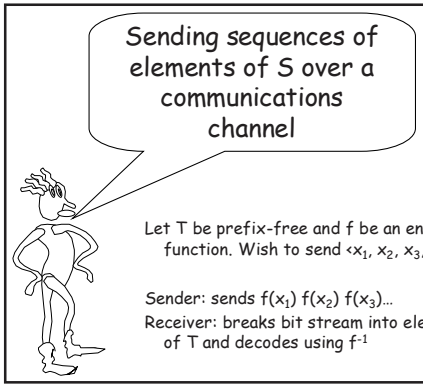
## Prefix-free Code for S

Let S be any set.
Definition: A prefix-free code for S is
a prefix-free set T and
a 1-1 "encoding" function f: S -> T.

The inverse function $f^{-1}$ is called the "decoding function".

Example: S = {apple, orange, mango}.
T = {0, 110, 1111}.
f(apple) = 0, f(orange) = 1111, f(mango) = 110.




Let T be prefix-free and f be an encoding function. Wish to send $\langle x_1, x_2, x_3, \ldots \rangle$

Sender: sends $f(x_1)\ f(x_2)\ f(x_3)\ldots$
Receiver: breaks bit stream into elements of T and decodes using $f^{-1}$

## Sending info on a channel

Example: S = {apple, orange, mango}.
T = {0, 110, 1111}.
f(apple) = 0, f(orange) = 1111, f(mango) = 110.

If we see
00011011111100…
we know it must be
0 0 0 110 1111 110 0 …
and hence
apple apple apple mango orange mango apple …

## Morse Code is not Prefix-free!

SOS encodes as …---…

| | | | | | |
|---|---|---|---|---|---|
| A .- | F ..-. | K -.- | P .--. | U ..- | Z --.. |
| B -... | G --. | L .-.. | Q --.- | V ...- | |
| C -.-. | H .... | M -- | R .-. | W .-- | |
| D -.. | I .. | N -. | S ... | X -..- | |
| E . | J .--- | O --- | T - | Y -.-- | |

## Morse Code is not Prefix-free!

SOS encodes as …---…

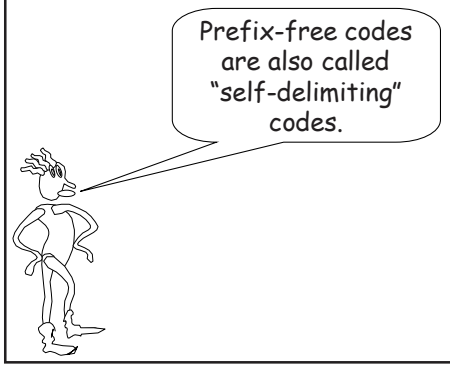
Could decode as: ..|.-|--|..|. = IAMIE

| | | | | | |
|---|---|---|---|---|---|
| A .- | F ..-. | K -.- | P .--. | U ..- | Z --.. |
| B -... | G --. | L .-.. | Q --.- | V ...- | |
| C -.-. | H .... | M -- | R .-. | W .-- | |
| D -.. | I .. | N -. | S ... | X -..- | |
| E . | J .--- | O --- | T - | Y -.-- | |

## Unless you use pauses

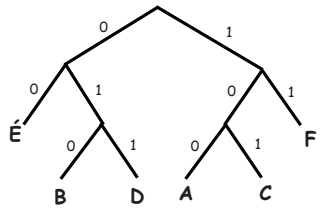
SOS encodes as … --- …

| | | | | | |
|---|---|---|---|---|---|
| A .- | F ..-. | K -.- | P .--. | U ..- | Z --.. |
| B -... | G --. | L .-.. | Q --.- | V ...- | |
| C -.-. | H .... | M -- | R .-. | W .-- | |
| D -.. | I .. | N -. | S ... | X -..- | |
| E . | J .--- | O --- | T - | Y -.-- | |

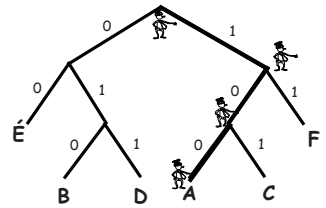



## Representing prefix-free codes

A = 100
B = 010
C = 101
D = 011
É = 00
F = 11

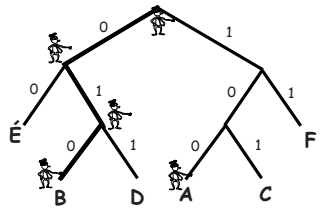



"CAFÉ" would encode as 1011001100
How do we decode 1011001100 (fast)?



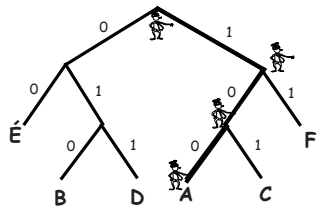

If you see: 10001010001110110011 00

can decode as:



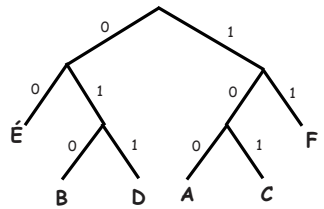

If you see: 100010100011101100 1100

can decode as: A



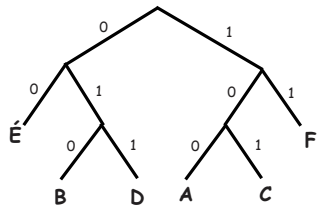

If you see: 100010100011101100 1100

can decode as: AB



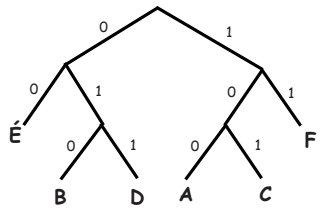

If you see: 100010100011101100 1100

can decode as: ABA



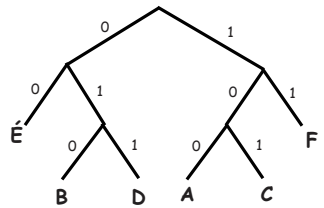

If you see: 100010100011101100 1100

can decode as: ABAD



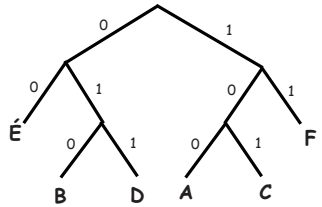

If you see: 100010100011101100 1100

can decode as: ABADC



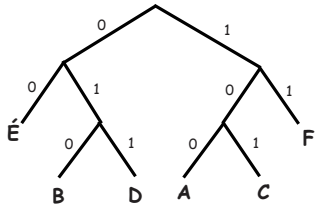

If you see: 100010100011101100 1100

can decode as: ABADCA



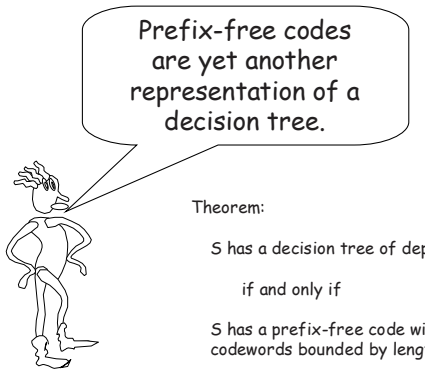

If you see: 1000101000111011001100

can decode as: ABADCAF




If you see: 1000101000111011001100

can decode as: ABADCAFÉ




Theorem:

S has a decision tree of depth d

if and only if

S has a prefix-free code with all codewords bounded by length d

Theorem:

S has a decision tree of depth d

if and only if

S has a prefix-free code with all codewords bounded by length d

## Extends to infinite sets

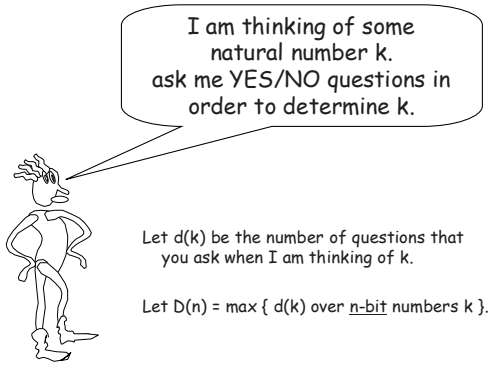
Let S is a subset of $\Sigma^*$

Theorem:

S has a decision tree where all length n elements of S have depth ≤ D(n)

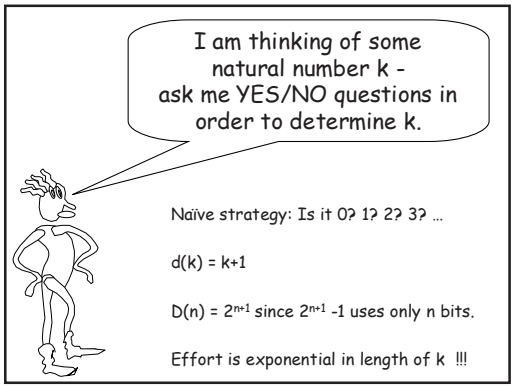
if and only if

S has a prefix-free code where all length n strings in S have encodings of length ≤ D(n)




Let d(k) be the number of questions that you ask when I am thinking of k.

Let D(n) = max { d(k) over n-bit numbers k }.

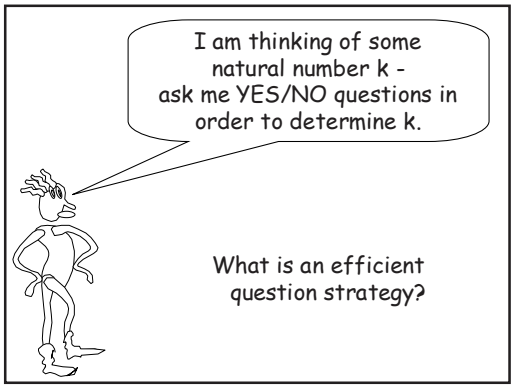



Naïve strategy: Is it 0? 1? 2? 3? …

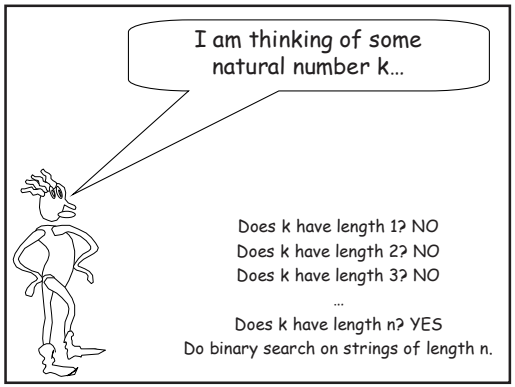
$d(k) = k+1$

$D(n) = 2^{n+1}$ since $2^{n+1} - 1$ uses only n bits.

Effort is exponential in length of k !!!




What is an efficient question strategy?



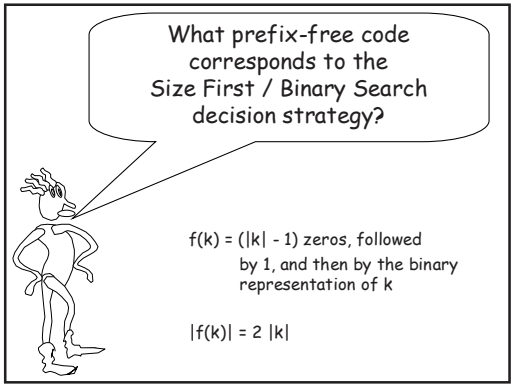

Does k have length 1? NO
Does k have length 2? NO
Does k have length 3? NO
…
Does k have length n? YES
Do binary search on strings of length n.

$d(k) = |k| + |k|$
$= 2 ( \lfloor \log k \rfloor + 1 )$

$D(n) = 2n$

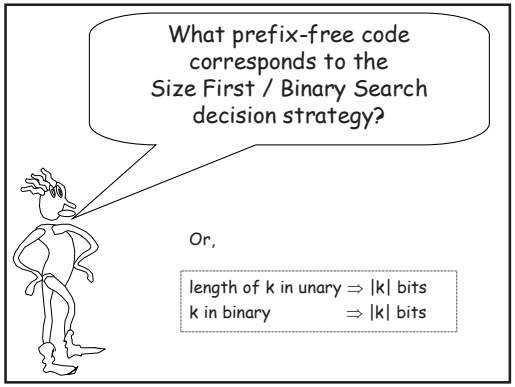
Size First/ Binary Search

Does k have length 1? NO
Does k have length 2? NO
Does k have length 3? NO
…
Does k have length n? YES
Do binary search on strings of length n.




$f(k)$ = ($|k| - 1$) zeros, followed by 1, and then by the binary representation of k

$|f(k)| = 2\,|k|$




Or,

length of k in unary $\Rightarrow$ $|k|$ bits
k in binary $\Rightarrow$ $|k|$ bits

## Another way to look at f

k = 27 = 11011, and hence |k| = 5

f(k) = 00001 11011

## Another way to look at f

k = 27 = 11011, and hence |k| = 5

11011

f(k) = 00001 11011

0101000111

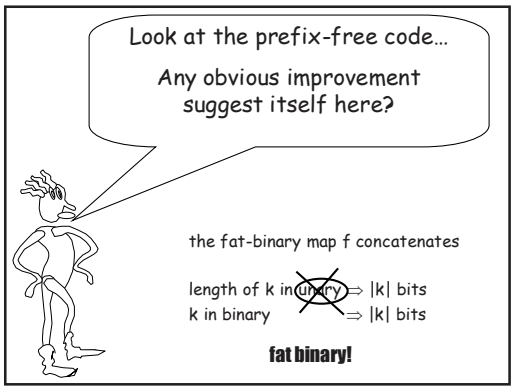
g(k) = 0101000111

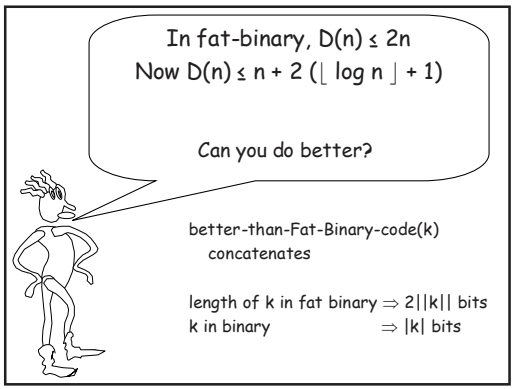
Another way to look at the function g:

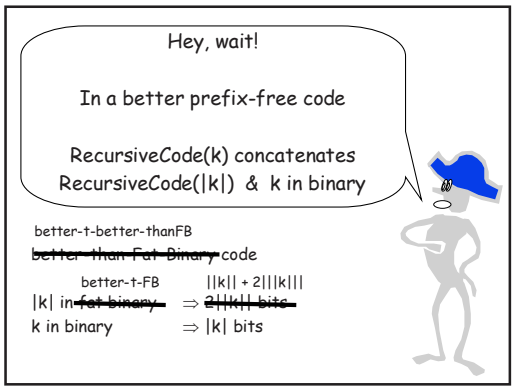
g(final 0) -> 10 g(all other 0's) -> 00
g(final 1) -> 11 g(all other 1's) -> 01

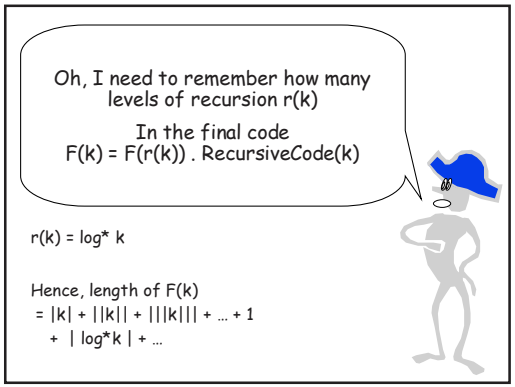
"Fat Binary" ⇔ Size First/Binary Search strategy

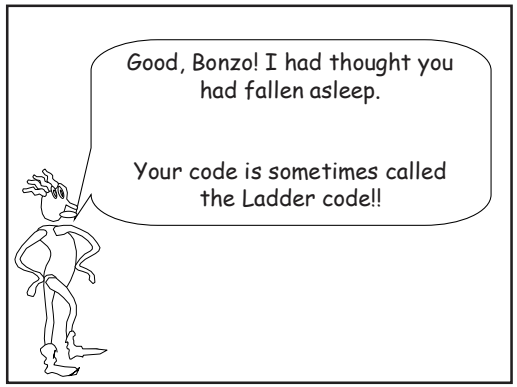
Is it possible to beat 2n questions to find a number of length n?

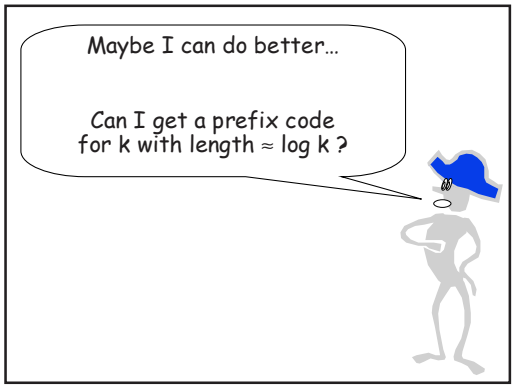


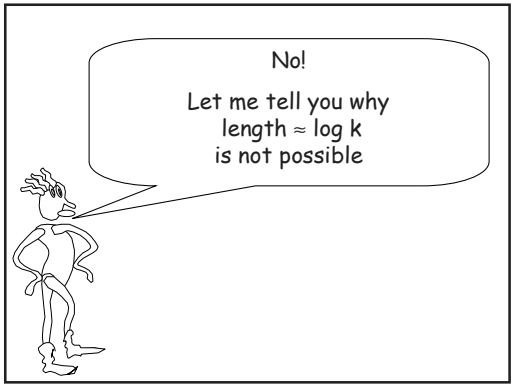



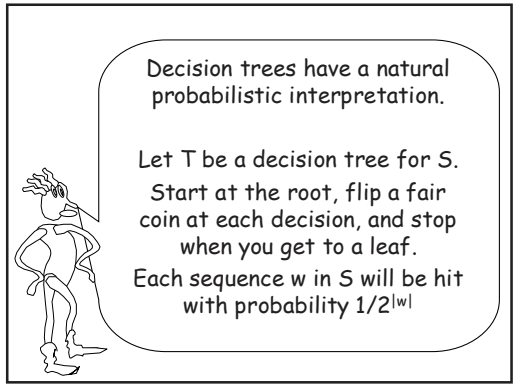



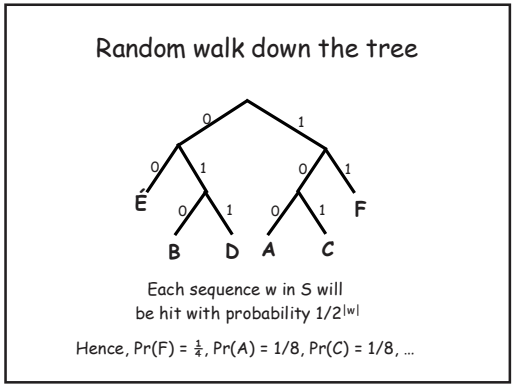

Oh, I need to remember how many levels of recursion r(k)
In the final code F(k) = F(r(k)) . RecursiveCode(k)
r(k) = log* k
Hence, length of F(k)
= |k| + ||k|| + |||k||| + … + 1
+ | log*k | + …

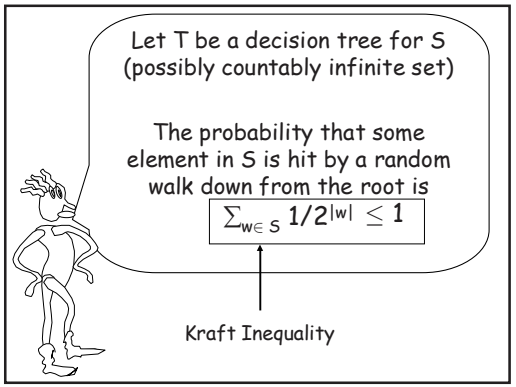

Good, Bonzo! I had thought you had fallen asleep.
Your code is sometimes called the Ladder code!!

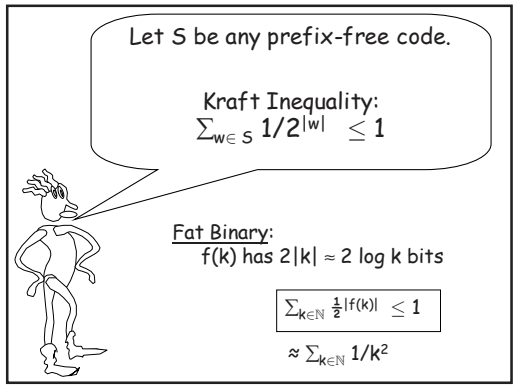

Maybe I can do better…
Can I get a prefix code for k with length ≈ log k ?

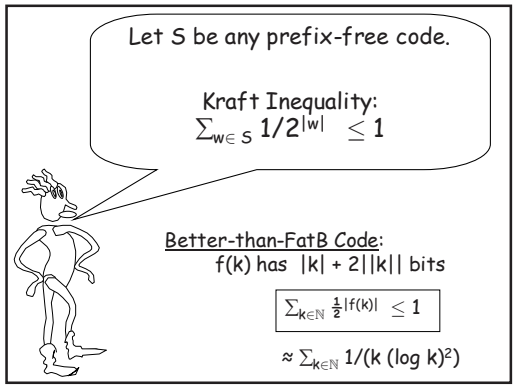

No!
Let me tell you why length ≈ log k is not possible

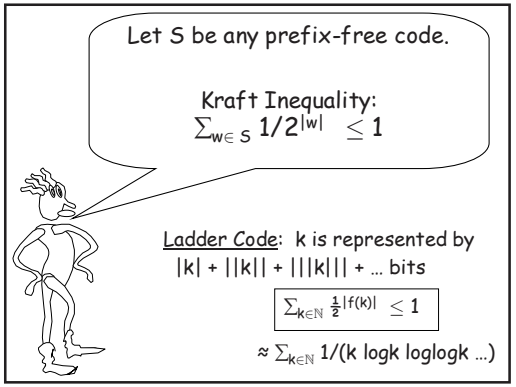

Decision trees have a natural probabilistic interpretation.
Let T be a decision tree for S.
Start at the root, flip a fair coin at each decision, and stop when you get to a leaf.
Each sequence w in S will be hit with probability 1/2^|w|

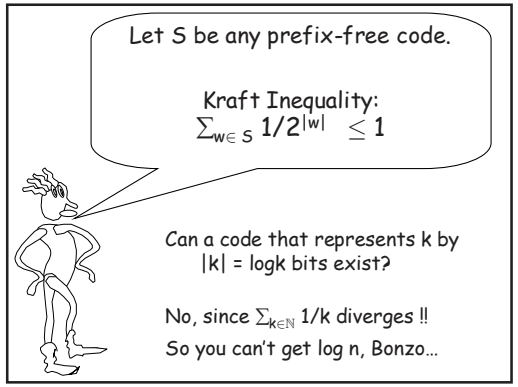

Random walk down the tree
0
1
0
1
0
1
E
0
1
0
1
F
B
D
A
C
Each sequence w in S will be hit with probability 1/2^|w|
Hence, Pr(F) = ¼, Pr(A) = 1/8, Pr(C) = 1/8, …

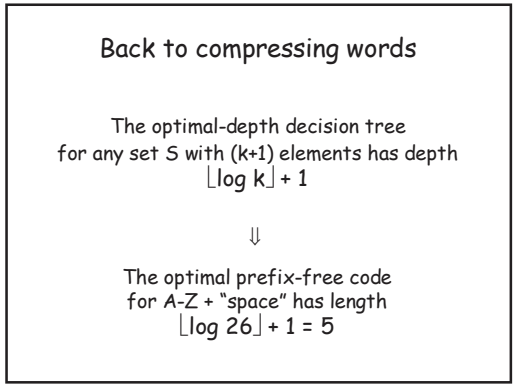

Let T be a decision tree for S
(possibly countably infinite set)
The probability that some element in S is hit by a random walk down from the root is
$\sum_{w \in S} 1/2^{|w|} \leq 1$
Kraft Inequality


Let S be any prefix-free code.
Kraft Inequality:
$\sum_{w \in S} 1/2^{|w|} \leq 1$
Fat Binary:
f(k) has $2|k| \approx 2 \log k$ bits
$\sum_{k \in \mathbb{N}} \frac{1}{2}^{|f(k)|} \leq 1$
$\approx \sum_{k \in \mathbb{N}} 1/k^2$


Let S be any prefix-free code.
Kraft Inequality:
$\sum_{w \in S} 1/2^{|w|} \leq 1$
Better-than-FatB Code:
f(k) has $|k| + 2||k||$ bits
$\sum_{k \in \mathbb{N}} \frac{1}{2}^{|f(k)|} \leq 1$
$\approx \sum_{k \in \mathbb{N}} 1/(k (\log k)^2)$


Let S be any prefix-free code.
Kraft Inequality:
$\sum_{w \in S} 1/2^{|w|} \leq 1$
Ladder Code: k is represented by
$|k| + ||k|| + |||k||| + \ldots$ bits
$\sum_{k \in \mathbb{N}} \frac{1}{2}^{|f(k)|} \leq 1$
$\approx \sum_{k \in \mathbb{N}} 1/(k \log k \log\log k \ldots)$


Let S be any prefix-free code.
Kraft Inequality:
$\sum_{w \in S} 1/2^{|w|} \leq 1$
Can a code that represents k by $|k| = \log k$ bits exist?
No, since $\sum_{k \in \mathbb{N}} 1/k$ diverges !!
So you can't get log n, Bonzo…


Back to compressing words
The optimal-depth decision tree for any set S with (k+1) elements has depth
$\lfloor \log k \rfloor + 1$
⇓
The optimal prefix-free code for A-Z + "space" has length
$\lfloor \log 26 \rfloor + 1 = 5$

## English Letter Frequencies

But in English, different letters occur with different *frequencies*.

| | | | | | |
|---|---|---|---|---|---|
| A 8.1% | F 2.3% | K .79% | P 1.6% | U 2.8% | Z .04% |
| B 1.4% | G 2.1% | L 3.7% | Q .11% | V .86% | |
| C 2.3% | H 6.6% | M 2.6% | R 6.2% | W 2.4% | |
| D 4.7% | I 6.8% | N 7.1% | S 6.3% | X .11% | |
| E 12% | J .11% | O 7.7% | T 9.0% | Y 2.0% | |

**ETAONIHSRDLUMWCFGYPBVKQXJZ**

## short encodings!

Why should we try to minimize the maximum length of a codeword?

If encoding A-Z, we will be happy if the "average codeword" is short.

## Morse Code

| | | | | | |
|---|---|---|---|---|---|
| A .- | F ..-. | K -.- | P .--. | U ..- | Z --.. |
| B -... | G --. | L .-.. | Q --.- | V ...- | |
| C -.-. | H .... | M -- | R .-. | W .-- | |
| D -.. | I .. | N -. | S ... | X -..- | |
| E . | J .--- | O --- | T - | Y -.-- | |

**ETAONIHSRDLUMWCFGYPBVKQXJZ**

Given frequencies for A-Z, what is the optimal prefix-free encoding of the alphabet?

I.e., one that minimizes the average code length

## Huffman Codes: Optimal Prefix-free Codes Relative to a Given Distribution

Here is a Huffman code based on the English letter frequencies given earlier:

| | | | | |
|---|---|---|---|---|
| A 1011 | F 101001 | K 10101000 | P 111000 | U 00100 |
| B 111001 | G 101000 | L 11101 | Q 1010100100 | V 1010101 |
| C 01010 | H 1100 | M 00101 | R 0011 | W 01011 |
| D 0100 | I 1111 | N 1000 | S 1101 | X 1010100101 |
| E 000 | J 1010100110 | O 1001 | T 011 | Y 101011 |
| | | | | Z 1010100111 |

But Huffman coding uses only letter frequencies.

For any fixed language, we can use correlations!
E.g., Q is almost always followed by U…

## Random words

Randomly generated letters from A-Z, space not using the frequencies at all:

XFOML RXKHRJFFJUJ ALPWXFWJXYJ
FFJEYVJCQSGHYD QPAAMKBZAACIBZLKJQD

## Random words

Using only single character frequencies:

OCRO HLO RGWR NMIELWIS EU LL NBNESEBYA TH EEI ALHENHTTPA OOBTTVA NAH BRL

## Random words

Each letter depends on the previous letter:

ON IE ANTSOUTINYS ARE T INCTORE ST BE S DEAMY ACHIN D ILONASIVE TUCOOWE AT TEASONARE FUSO TIZIN ANDY TOBE SEACE CTISBE

## Random words

Each letter depends on 2 previous letters:

IN NO IST LAT WHEY CRATICT FROURE BIRS GROCID PONDENOME OF DEMONSTURES OF THE REPTAGIN IS REGOACTIONA OF CRE

## Random words

Each letter depends on 3 previous letters:

THE GENERATED JOB PROVIDUAL BETTER TRAND THE DISPLAYED CODE, ABOVERY UPONDULTS WELL THE CODERST IN THESTICAL IT DO HOCK BOTHEMERG.

(INSTATES CONS ERATION. NEVER ANY OF PUBLE AND TO THEORY. EVENTIAL CALLEGAND TO ELAST BENERATED IN WITH PIES AS IS WITH THE)

## References

*The Mathematical Theory of Communication,* by C. Shannon and W. Weaver

*Elements of Information Theory,* by T. Cover and J. Thomas