Great Theoretical Ideas In Computer Science
Steven Rudich, Anupam Gupta — CS 15-251 — Spring 2005
Lecture 21 — March 29, 2005 — Carnegie Mellon University

Today, we will learn about a formidable tool in probability that will allow us to solve problems that seem really really messy…

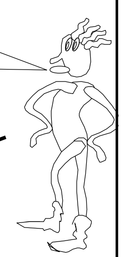
If I randomly put 100 letters into 100 addressed envelopes, on average how many letters will end up in their correct envelopes?

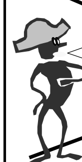
~~Hmm…~~

~~$\sum_k k \cdot \Pr($*exactly* k letters end up in correct envelopes$)$~~

~~$= \sum_k k \cdot (\ldots$aargh!!…$)$~~

On average, in class of size m, how many pairs of people will have the same birthday?



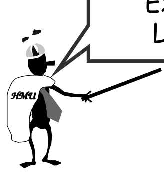


~~$\sum_k k \cdot \Pr($exactly k collisions$)$~~

~~$= \sum_k k \cdot (\ldots$aargh!!!!…$)$~~

The new tool is called "Linearity of Expectation"

Expectatus
Linearitus



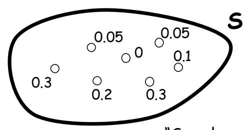
Random Variable

To use this new tool, we will also need to understand the concept of a Random Variable

Today's goal: not too much material, but to understand it well.

Probability Distribution

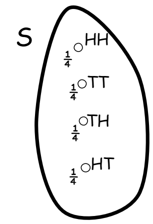
A (finite) probability distribution D
- a finite set S of elements (samples)
- each $x \in S$ has weight or probability $p(x) \in [0,1]$



weights must sum to 1

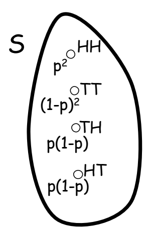
"Sample space"

Flip penny and nickel (unbiased)

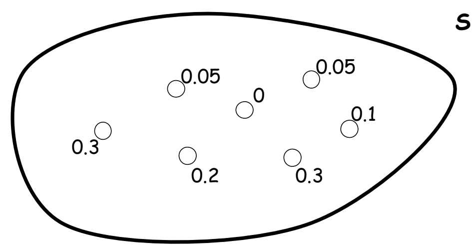


Flip penny and nickel (biased)

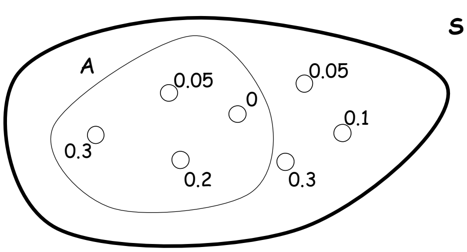
heads probability = p



Probability Distribution



An event is a subset



$\Pr[A] = \sum_{x \in A} p(x) = 0.55$

Running Example

I throw a white die and a black die.

Sample space S =

{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
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(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }

$\Pr(x) = 1/36$
$\forall x \in S$

E = event that sum ≤ 3

$\Pr[E] = |E|/|S|$ = proportion of E in S = 3/36

New concept: <u>Random Variables</u>

Random Variables

Random Variable: a (real-valued) function on S

Toss a white die and a black die.

Sample space S =
{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
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Examples:
- X = value of white die.
  - X(3,4) = 3, X(1,6) = 1 etc.
- Y = sum of values of the two dice.
  - Y(3,4) = 7, Y(1,6) = 7 etc.
- W = (value of white die)$^{\text{value of black die}}$
  - W(3,4) = 3^4 Y(1,6) = 1^6
- Z = (1 if two dice are equal, 0 otherwise)
  - Z(4,4) = 1, Z(1,6) = 0 etc.

E.g., tossing a fair coin n times

S = all sequences of $\{H, T\}^n$

D = uniform distribution on S
$\Rightarrow D(x) = (\frac{1}{2})^n$ for all $x \in S$

Random Variables (say n = 10)
- X = # of heads
  - X(HHHTTHTHTT) = 5
- Y = (1 if #heads = #tails, 0 otherwise)
  - Y(HHHTTHTHTT) = 1, Y(THHHHTTTTT) = 0

Notational conventions

Use letters like A, B, E for events.

Use letters like X, Y, f, g for R.V.'s.

R.V. = random variable

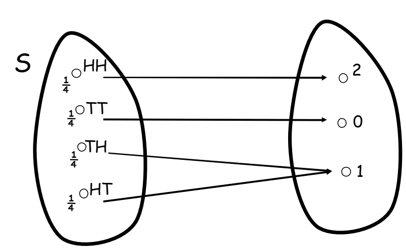
Two views of random variables

Think of a R.V. as

- a function from S to the reals $\mathbb{R}$
- or think of the induced distribution on $\mathbb{R}$

Two coins tossed

X: {TT, TH, HT, HH} -> {0, 1, 2}
counts the number of heads



Two views of random variables

Think of a R.V. as

- $\boxed{\text{a function from S to the reals P}}$

- or think of the induced distribution on P

Two coins tossed

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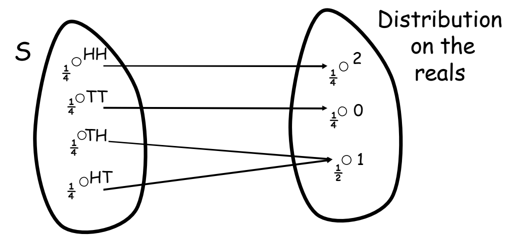

Two views of random variables

Think of a R.V. as

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Two dice

I throw a white die and a black die.

Sample space S =
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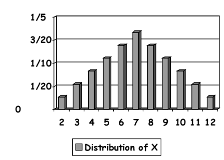
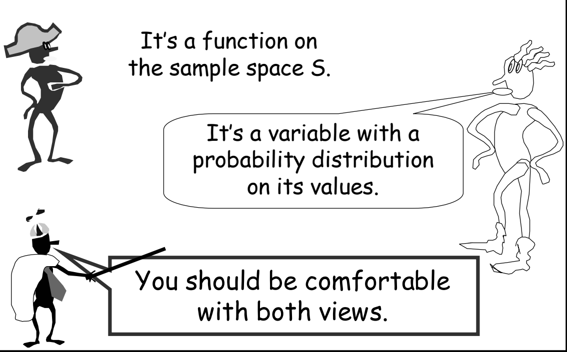


X = sum of both dice

function with X(1,1) = 2, X(1,2)=X(2,1)=3, …, X(6,6)=12

It's a floor wax *and* a dessert topping



From Random Variables to Events

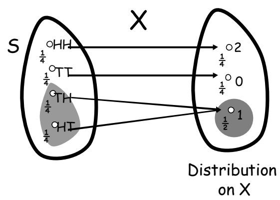
For any random variable X and value a,
we can define the <u>event</u> A that X=a.

Pr(A) = Pr(X=a) = Pr({x ∈ S| X(x)=a}).

Two coins tossed

X: {TT, TH, HT, HH} -> {0, 1, 2}
counts the number of heads

$\boxed{\Pr(X = a) = \Pr(\{x \in S \mid X(x) = a\})}$



Pr(X = 1)

= $\Pr(\{x \in S \mid X(x) = 1\})$

= Pr({TH, HT}) = ½.

Two dice

I throw a white die and a black die. X = sum

Sample space S =
{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
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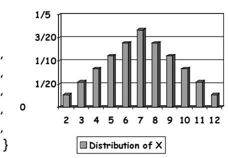


Pr(X = 6)
= $\Pr(\{x \in S \mid X(x) = 6\})$
= 5/36.

$\boxed{\Pr(X = a) = \Pr(\{x \in S \mid X(x) = a\})}$

From Random Variables to Events

For any random variable X and value a,
we can define the <u>event</u> A that X=a.

$\Pr(A) = \Pr(X=a) = \Pr(\{x \in S \mid X(x)=a\})$.

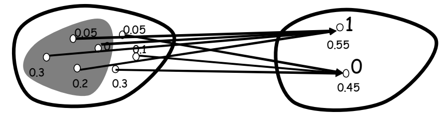
X has a distribution on its values

X is a function on the sample space S

From Events to Random Variables

For any event A, can define the <u>indicator random variable</u> for A:

$$X_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$



Definition: expectation

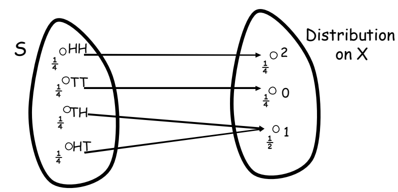
The <u>expectation</u>, or <u>expected value</u> of a random variable X is written as E[X], and is

$$E[X] = \sum_{x \in S} \Pr(x) X(x) = \sum_k k \Pr(X = k)$$

X is a function on the sample space S

X has a distribution on its values

Thinking about expectation



$\boxed{\sum_{x \in S} X(x) \Pr(x)}$ $\boxed{\sum_k k \Pr(X = k)}$

E[X] = ¼*0 + ¼*1 + ¼*1 + ¼*2 = 1.

E[X] = ¼*0 + ½*1 + ¼*2 = 1.

A quick calculation…

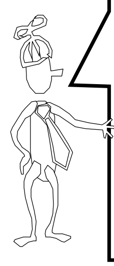
What if I flip a coin 3 times? Now what is the expected number of heads?

$E[X] = (1/8) \times 0 + (3/8) \times 1 + (3/8) \times 2 + (1/8) \times 3 = 1.5$

But Pr[X = 1.5] = 0…

Moral: don't always expect the expected.
Pr[X = E[X]] may be 0 !

Type checking



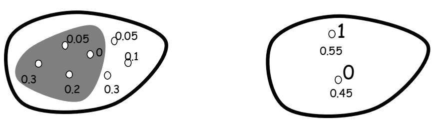
A Random Variable is the type of thing you might want to know an expected value of.

If you are computing an expectation, the thing whose expectation you are computing is a random variable.

Indicator R.V.s: $E[X_A] = Pr(A)$

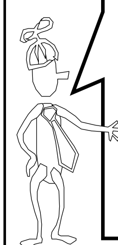
For event A, the <u>indicator random variable</u> for A:

$$X_A(x) = 1 \quad \text{if } x \in A$$
$$= 0 \quad \text{if } x \notin A$$



$\mathbf{E[X_A]} = 1 \times Pr(X_A = 1) = Pr(A)$

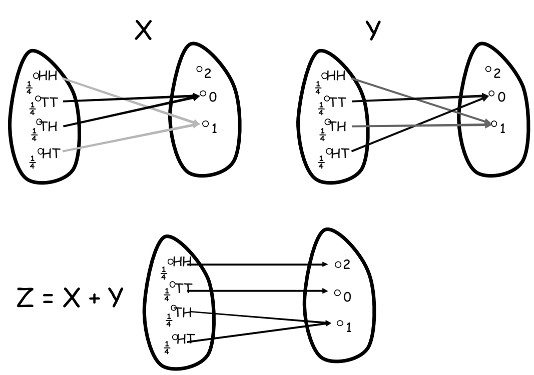
Adding Random Variables



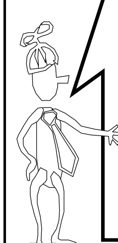
If X and Y are random variables (on the same set S), then Z = X + Y is also a random variable.

$$Z(x) \equiv X(x) + Y(x)$$

E.g., rolling two dice.
X = 1st die, Y= 2nd die,
Z = sum of two dice.



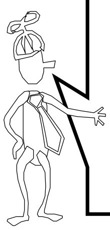
Adding Random Variables



Example: Consider picking a random person in the world. Let X = length of the person's left arm in inches. Y = length of the person's right arm in inches. Let Z = X+Y. Z measures the combined arm lengths.

Formally, S = {people in world}, D = uniform distribution on S.

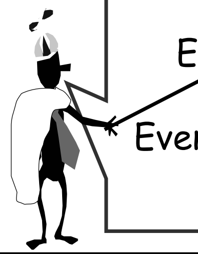
Independence



Two random variables X and Y are <u>independent</u> if for every a,b, the events X=a and Y=b are independent.

How about the case of
X=1[st] die, Y=2[nd] die?
X = left arm, Y=right arm?

Linearity of Expectation

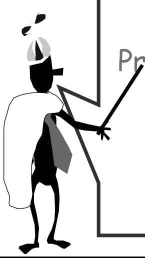


If Z = X+Y, then

E[Z] = E[X] + E[Y]

Even if X and Y are not independent.

Linearity of Expectation



If Z = X+Y, then

E[Z] = E[X] + E[Y]

Proof:

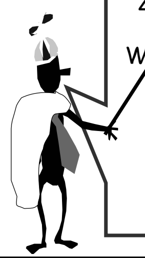
$E[X] = \sum_{x\in S} Pr(x)\, X(x)$

$E[Y] = \sum_{x\in S} Pr(x)\, Y(x)$

$E[Z] = \sum_{x\in S} Pr(x)\, Z(x)$

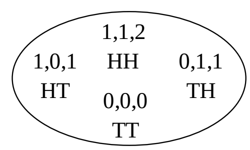
but $Z(x) = X(x) + Y(x)$

Linearity of Expectation

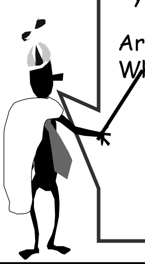


E.g., 2 fair flips:
X = 1[st] coin, Y = 2[nd] coin.
Z = X+Y = total # heads.

What is E[X] ? E[Y] ? E[Z] ?

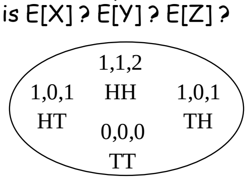


Linearity of Expectation



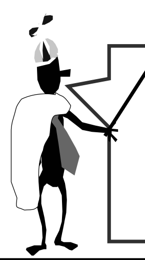
E.g., 2 fair flips:
X = at least one coin heads,
Y = both coins are heads, Z = X+Y

Are X and Y independent?
What is E[X] ? E[Y] ? E[Z] ?



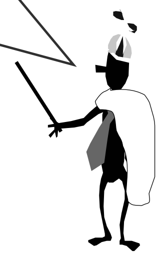
By induction

$E[X_1 + X_2 + \ldots + X_n] =$
$E[X_1] + E[X_2] + \ldots. + E[X_n]$

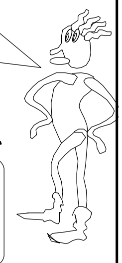


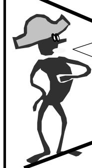
the expectation of the sum = the sum of the expectations

It is finally time to show off our probability prowess…



If I randomly put 100 letters into 100 addressed envelopes, on average how many letters will end up in their correct envelopes?

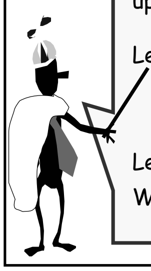




~~Hmm…~~

~~$\sum_k$ k·Pr(*exactly* k letters end up in correct envelopes)~~
~~$= \sum_k$ k· (…aargh!!…)~~

Use Linearity of Expectation



Let A_i be the event the i^{th} letter ends up in its correct envelope.

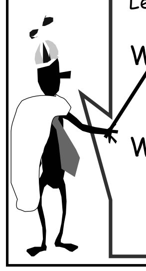
Let X_i be the indicator R.V. for A_i.

$$X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Let $Z = X_1 + \ldots + X_{100}$.

We are asking for E[Z].

Use Linearity of Expectation



Let A_i be the event the i^{th} letter ends up in the correct envelope.
Let X_i be the indicator R.V. for A_i.
Let $Z = X_1 + \ldots + X_n$. We are asking for E[Z].

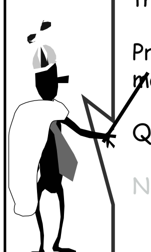
What is $E[X_i]$?

$E[X_i] = \Pr(A_i) = 1/100$.

What is E[Z]?

$$\begin{aligned} E[Z] &= E[X_1 + \ldots + X_{100}] \\ &= E[X_1] + \ldots + E[X_{100}] \\ &= 1/100 + \ldots + 1/100 = 1. \end{aligned}$$

Use Linearity of Expectation



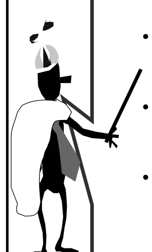
So, in expectation, 1 card will be in the same position as it started.

Pretty neat: it doesn't depend on how many cards!

Question: were the X_i independent?

No! E.g., think of n=2.

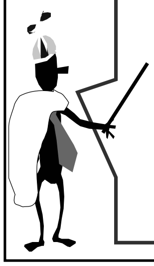
Use Linearity of Expectation



General approach:

- View thing you care about as expected value of some RV.
- Write this RV as sum of simpler RVs (typically indicator RVs).
- Solve for their expectations and add them up!

Example



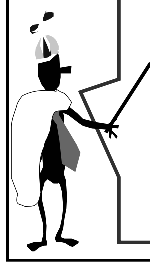
We flip n coins of bias p. What is the expected number of heads?

We could do this by summing

$\sum_k k \Pr(X = k)$

$= \sum_k k \binom{n}{k} p^k (1-p)^{n-k}$

But now we know a better way

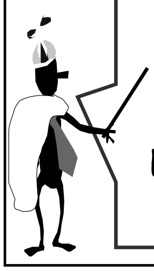


Let X = number of heads when n independent coins of bias p are flipped.

Break X into n simpler RVs,

$$X_i = \begin{cases} 0, & \text{if the } i^{th} \text{ coin is tails} \\ 1, & \text{if the } i^{th} \text{ coin is heads} \end{cases}$$

$$E[\ X\ ] = E[\ \Sigma_i X_i\ ] = ?$$



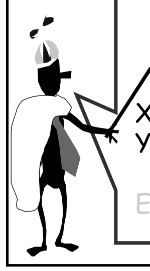
Let X = number of heads when n independent coins of bias p are flipped.

Break X into n simpler RVs,

$$X_i = \begin{cases} 0, & \text{if the } i^{th} \text{ coin is tails} \\ 1, & \text{if the } i^{th} \text{ coin is heads} \end{cases}$$

$$E[\ X\ ] = E[\ \Sigma_i X_i\ ] = np$$

What about Products?



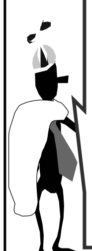
If Z = XY, then
$E[Z] = E[X] \times E[Y]$?

No!

X=indicator for "1^{st} flip is heads"
Y=indicator for "1^{st} flip is tails".

E[XY]=0.

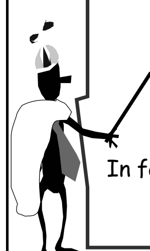
But it is true if RVs are independent



Proof:

$E[X] = \sum_a a \times \Pr(X=a)$
$E[Y] = \sum_b b \times \Pr(Y=b)$

$E[XY] = \sum_c c \times \Pr(XY = c)$
$= \sum_c \sum_{a,b:ab=c} c \times \Pr(X=a \cap Y=b)$
$= \sum_{a,b} ab \times \Pr(X=a \cap Y=b)$
$= \sum_{a,b} ab \times \Pr(X=a) \Pr(Y=b)$
$= E[X]\, E[Y]$



E.g., 2 fair flips.
X = indicator for 1^{st} coin being heads,
Y = indicator for 2^{nd} coin being heads.
XY = indicator for "both are heads".

$E[X] = \frac{1}{2}$, $E[Y] = \frac{1}{2}$, $E[XY] = \frac{1}{4}$.

$E[X*X] = E[X]^2$?

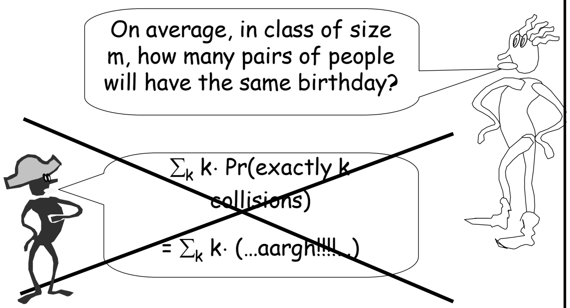
No: $E[X^2] = \frac{1}{2}$, $E[X]^2 = \frac{1}{4}$.

In fact, $E[X^2] - E[X]^2$ is called the *variance* of X.

Most of the time, though, power will come from using sums.

Mostly because Linearity of Expectations holds even if RVs are <u>not independent</u>.

Another problem



Use linearity of expectation.

Suppose we have m people each with a uniformly chosen birthday from 1 to 366.

X = number of pairs of people with the same birthday.

E[X] = ?

X = number of pairs of people with the same birthday.
E[X] = ?

Use m(m-1)/2 indicator variables, one for each pair of people.

X_{jk} = 1 if person j and person k have the same birthday; else 0.

$E[X_{jk}] = (1/366)\ 1 + (1 - 1/366)\ 0$
$= 1/366$

X = number of pairs of people with the same birthday.

$E[X] = E[\ \Sigma_{j \le k \le m} X_{jk}\ ]$

There are many dependencies among the indicator variables. E.g., X_{12} and X_{13} and X_{23} are dependent.

But we don't care!

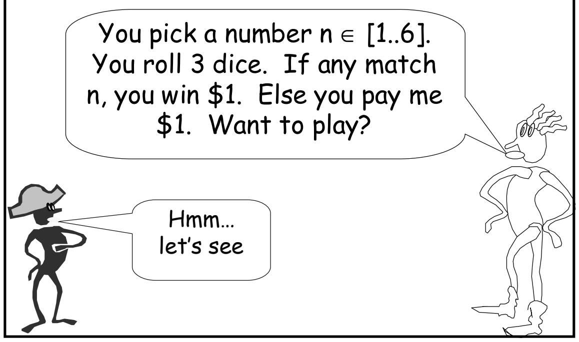
X = number of pairs of people with the same birthday.

$E[X] = E[\ \Sigma_{j \le k \le m} X_{jk}\ ]$

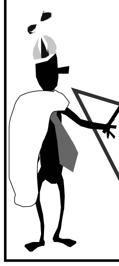
$= \Sigma_{j \le k \le m} E[\ X_{jk}\ ]$

$= m(m-1)/2 \times 1/366$

Step right up…



Analysis

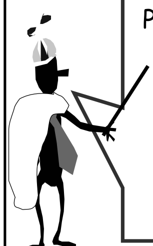


A_i = event that i-th die matches.
X_i = indicator RV for A_i.

Expected number of dice that match:
$E[X_1+X_2+X_3] = 1/6+1/6+1/6 = \frac{1}{2}$.

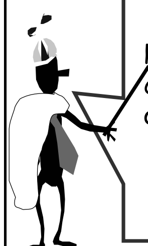
But this is not the same as Pr(at least one die matches).

Analysis



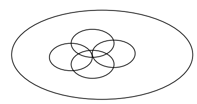
Pr(at least one die matches)
= 1 – Pr(none match)
= $1 - (5/6)^3 = 0.416$.

What's going on?

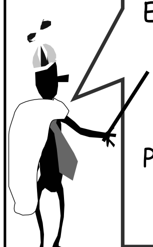


Say we have a collection of events A_1, A_2, ….

How does E[# events that occur] compare to Pr(at least one occurs)?



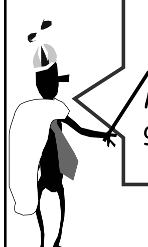
What's going on?



E[# events that occur]
= $\sum_k$ Pr(k events occur) × k
= $\sum_{(k > 0)}$ Pr(k events occur) × k

Pr(at least one event occurs)
= $\sum_{(k > 0)}$ Pr(k events occur)

What's going on?



Moral #1: be careful you are modeling problem correctly.

Moral #2: watch out for carnival games.