Great Theoretical Ideas In Computer Science

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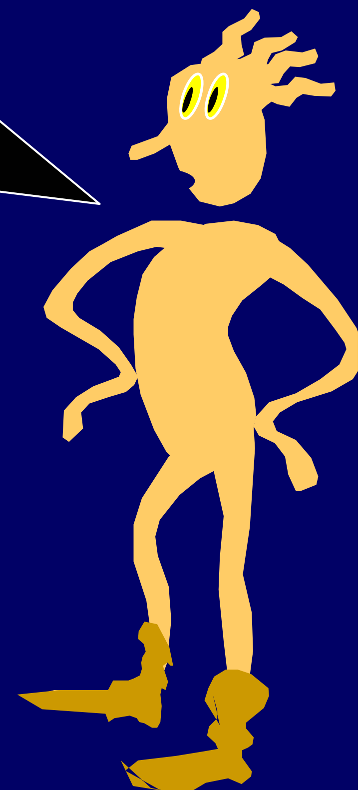
Today, we will learn about a formidable tool in probability that will allow us to solve problems that seem <u>really really messy</u>…



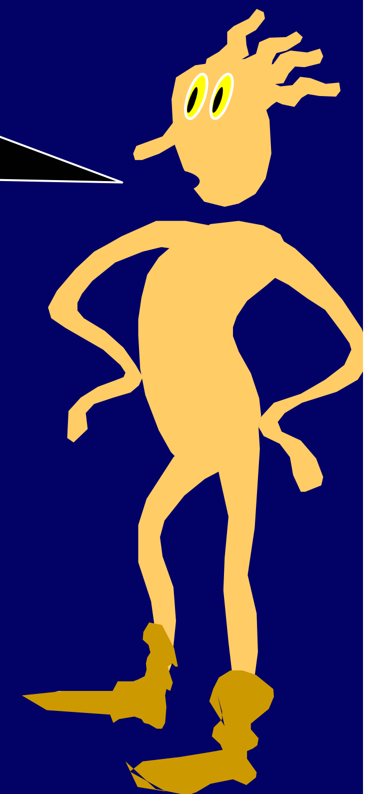
If I randomly put 100 letters into 100 addressed envelopes, <u>on average</u> how many letters will end up in their correct envelopes?

Hmm…

$\sum_k$ k·Pr(*exactly* k letters end up in correct envelopes)

= $\sum_k$ k· (…aargh!!…)

On average, in class of size m, how many pairs of people will have the same birthday?

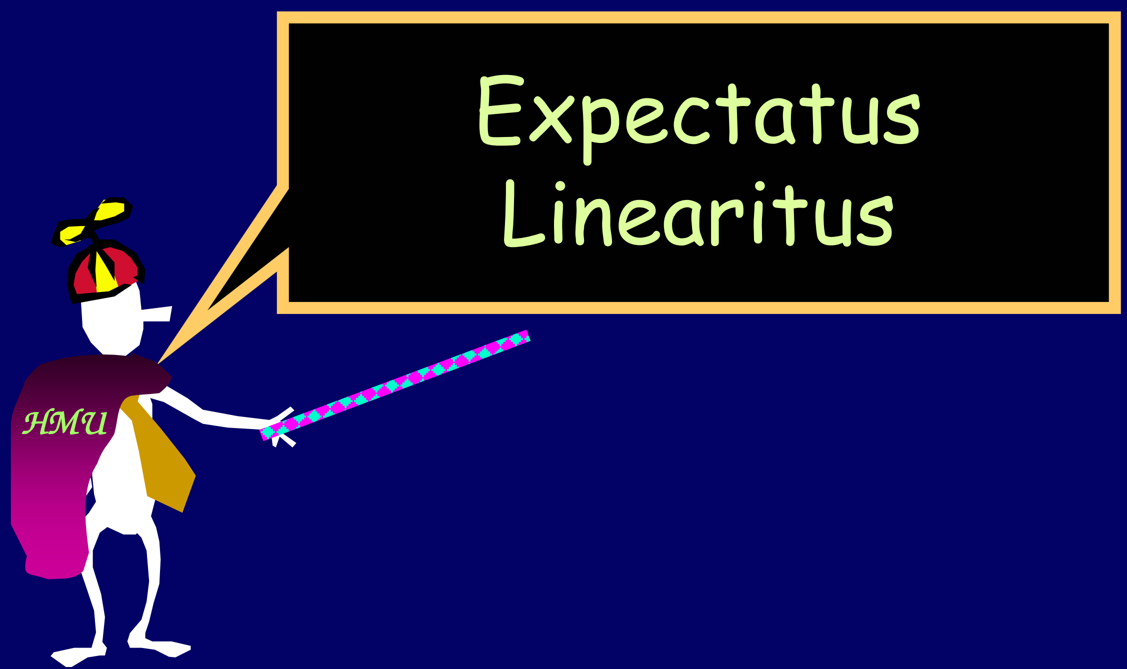


~~$\sum_k k\cdot$ Pr(exactly k collisions)~~

~~$= \sum_k k\cdot$ (...aargh!!!!...)~~

The new tool is called

"Linearity of Expectation"



Random Variable

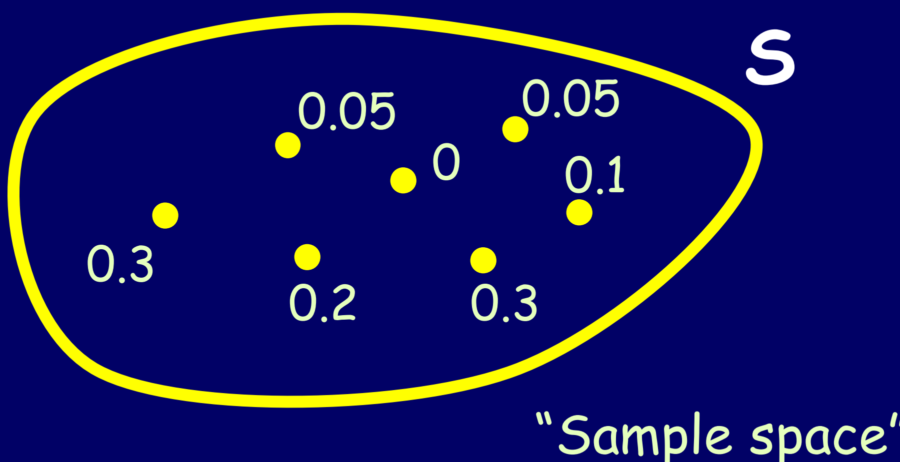
To use this new tool, we will also need to understand the concept of a <u>Random Variable</u>

Today's *goal*: not too much material, but to understand it well.

Probability Distribution

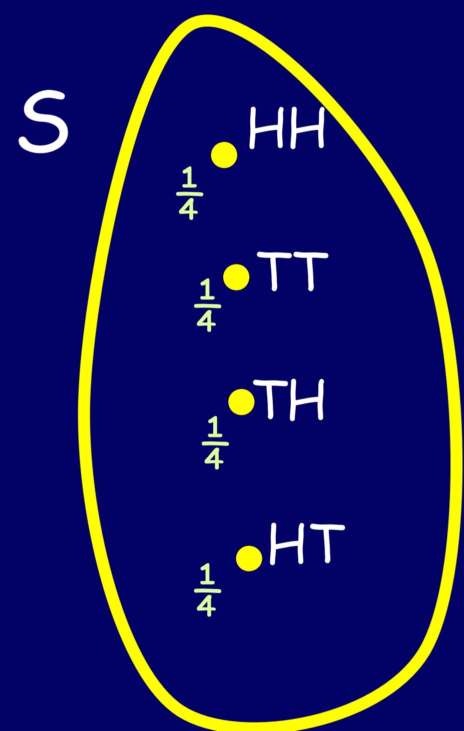
A (finite) probability distribution D

- a finite set S of elements (samples)
- each $x \in S$ has weight or probability $p(x) \in [0,1]$



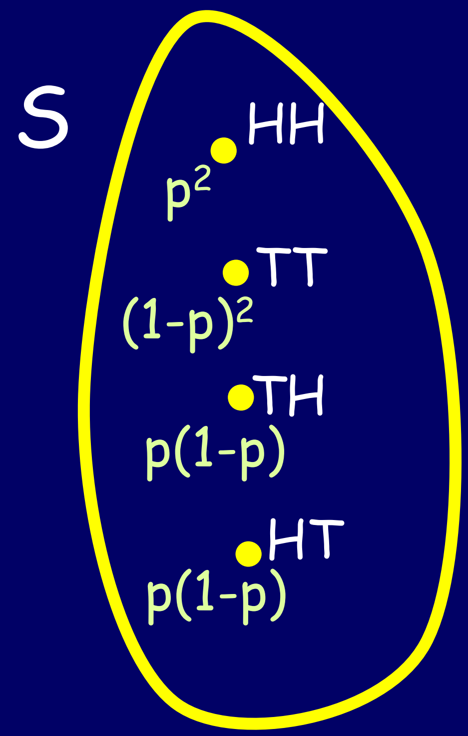
weights must sum to 1

Flip penny and nickel (unbiased)

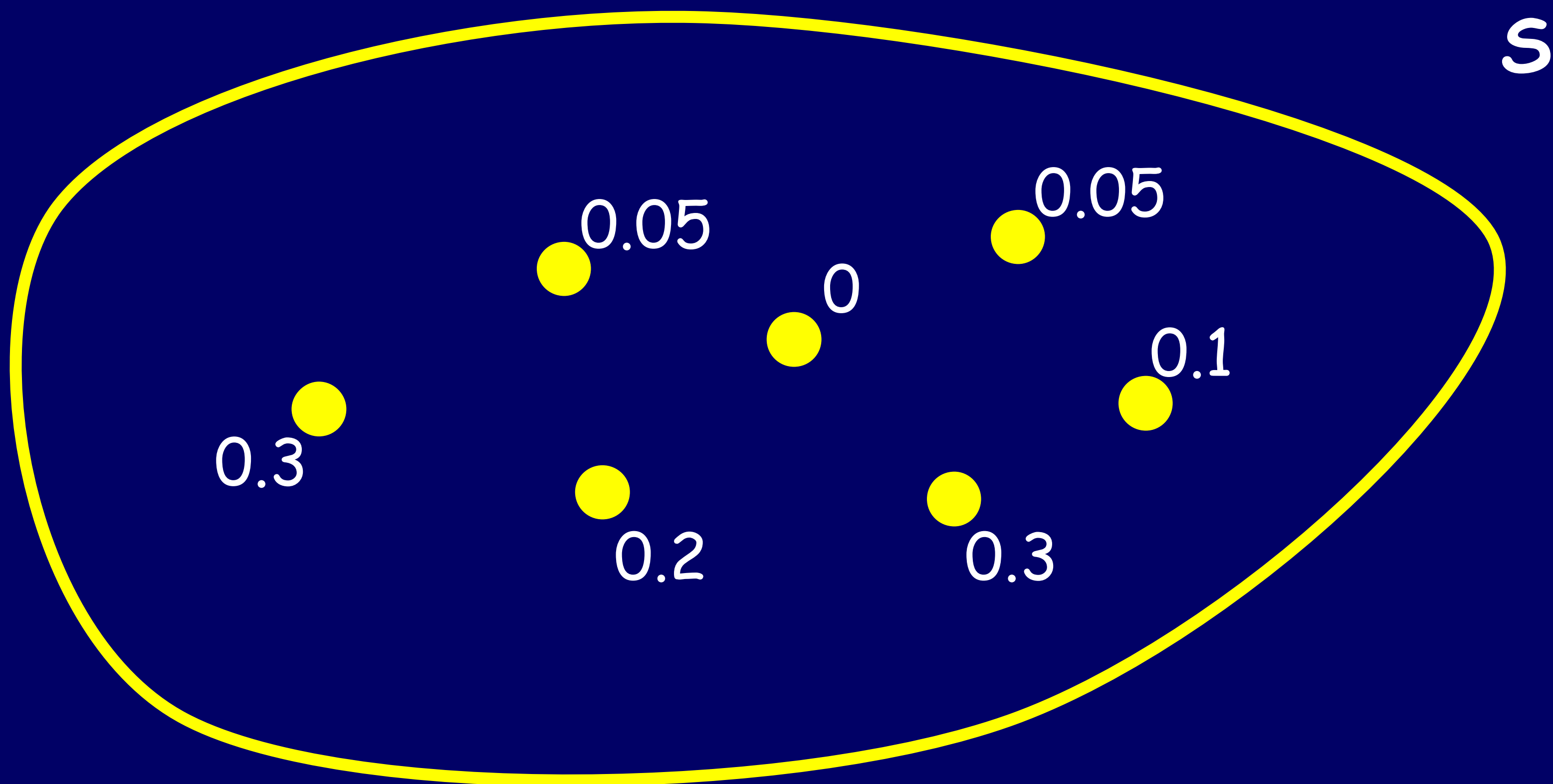


Flip penny and nickel (biased)

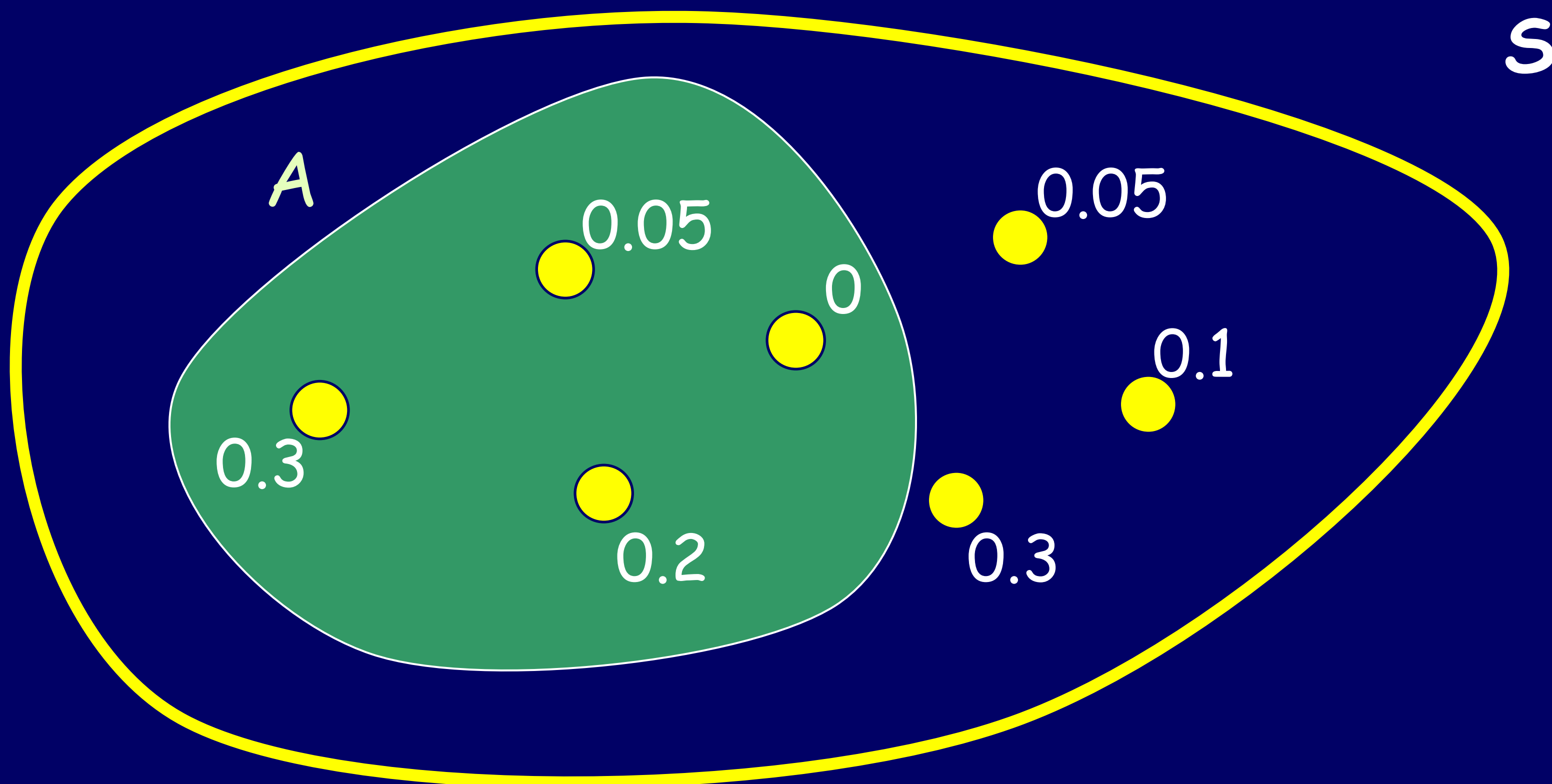
heads probability = p



Probability Distribution



An event is a subset



$$\Pr[A] = \sum_{x \in A} p(x) = 0.55$$

Running Example

I throw a white die and a black die.

Sample space S =

{ (1,1),	(1,2),	(1,3),	(1,4),	(1,5),	(1,6),
(2,1),	(2,2),	(2,3),	(2,4),	(2,5),	(2,6),
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(6,1),	(6,2),	(6,3),	(6,4),	(6,5),	(6,6) }

Pr(x) = 1/36
$\forall\, x \in S$

E = event that sum ≤ 3

Pr[E] = |E|/|S| = proportion of E in S = 3/36

New concept:
<u>Random Variables</u>

Random Variables

Random Variable: a (real-valued) function on S

Examples:

X = value of white die.

$X(3,4) = 3$, $X(1,6) = 1$ etc.

Y = sum of values of the two dice.

$Y(3,4) = 7$, $Y(1,6) = 7$ etc.

W = (value of white die)$^{\text{value of black die}}$

$W(3,4) = 3^4$ $Y(1,6) = 1^6$

Z = (1 if two dice are equal, 0 otherwise)

$Z(4,4) = 1$, $Z(1,6) = 0$ etc.

Toss a white die and a black die.

Sample space S =

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E.g., tossing a fair coin n times

S = all sequences of $\{H, T\}^n$

D = uniform distribution on S

$\Rightarrow D(x) = (\frac{1}{2})^n$ for all $x \in S$

Random Variables (say n = 10)

X = # of heads

X(HHHTTHTHTT) = 5

Y = (1 if #heads = #tails, 0 otherwise)

Y(HHHTTHTHTT) = 1, Y(THHHHTTTTT) = 0

Notational conventions

Use letters like A, B, E for events.

Use letters like X, Y, f, g for R.V.'s.

R.V. = random variable

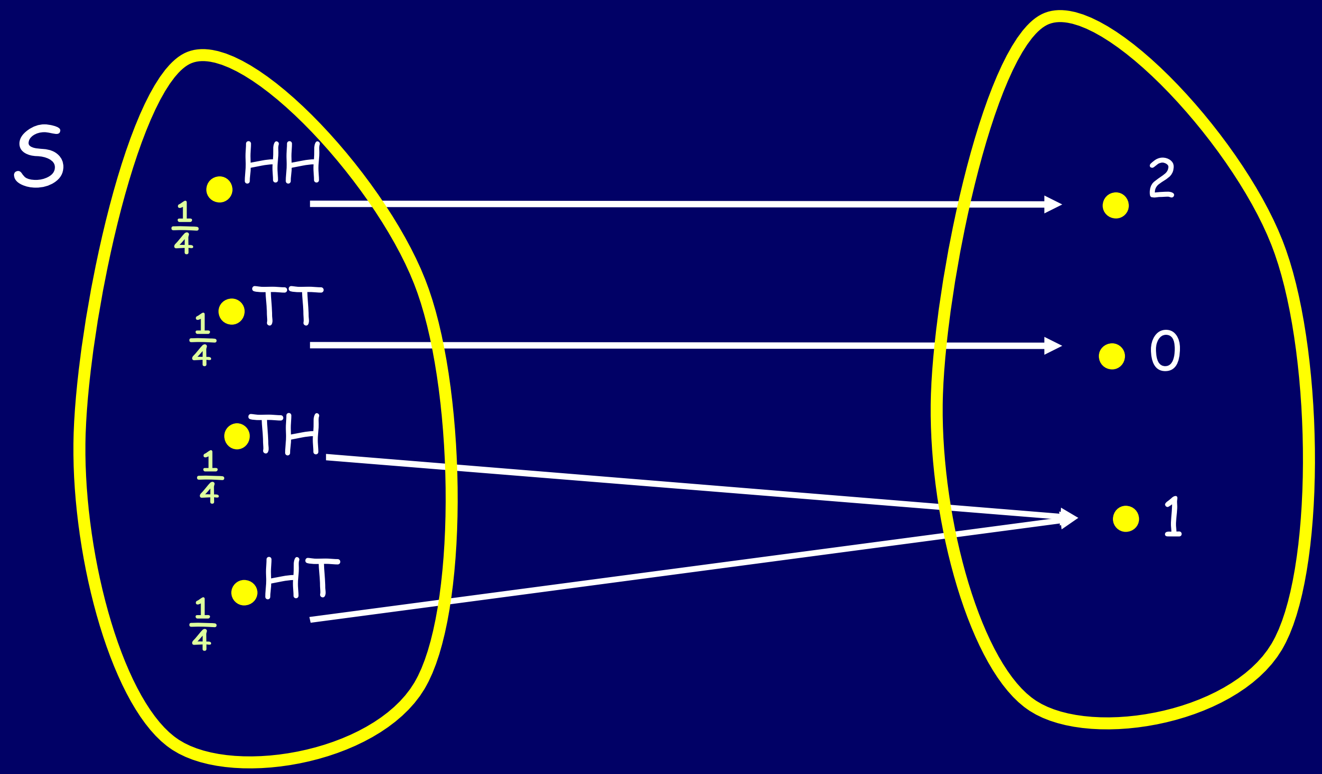
Two views of random variables

Think of a R.V. as

- a function from S to the reals P
- or think of the induced distribution on P

Two coins tossed

X: {TT, TH, HT, HH} -> {0, 1, 2}
counts the number of heads



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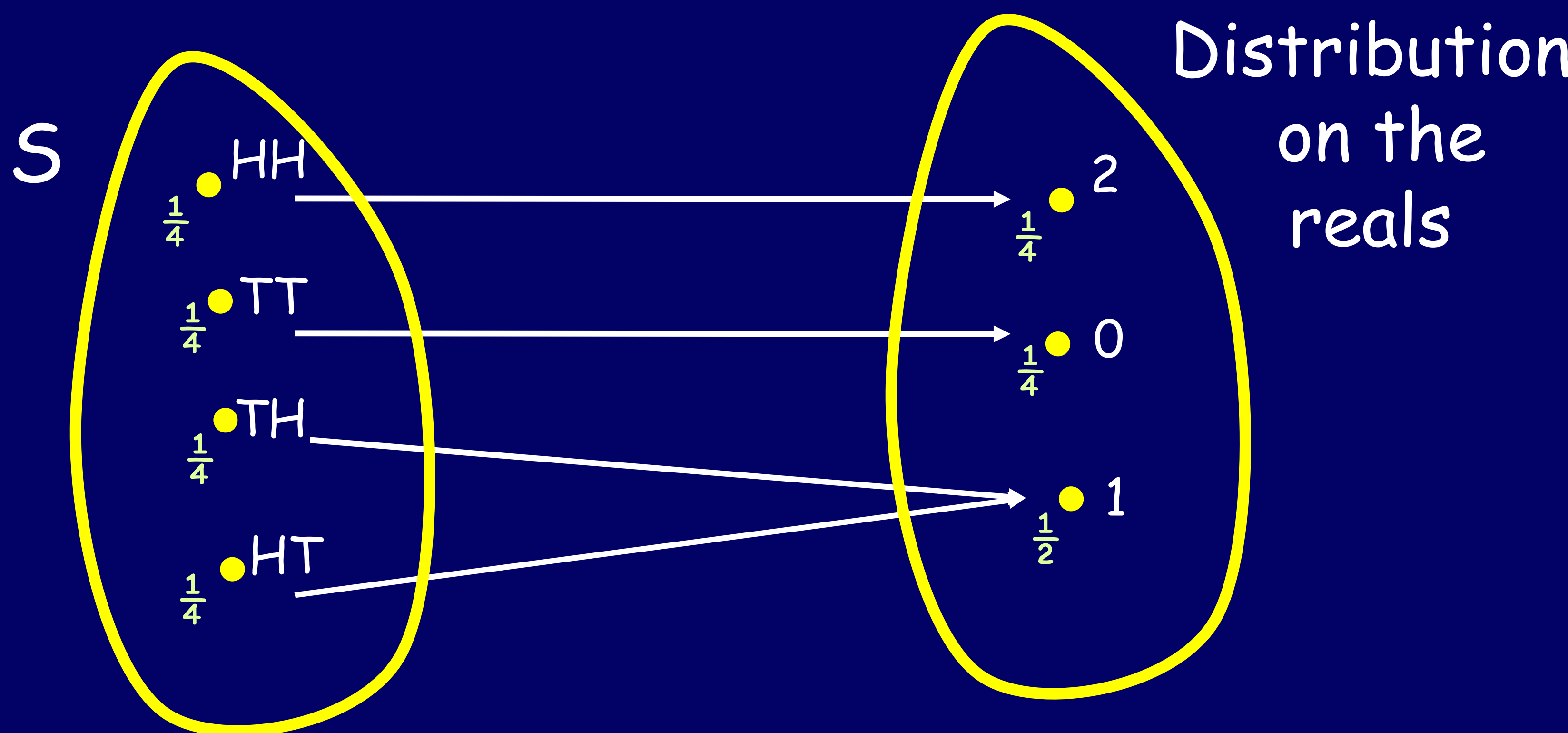

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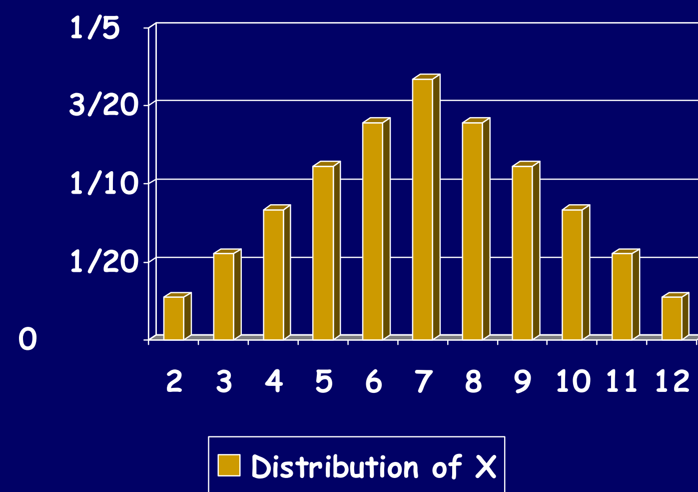
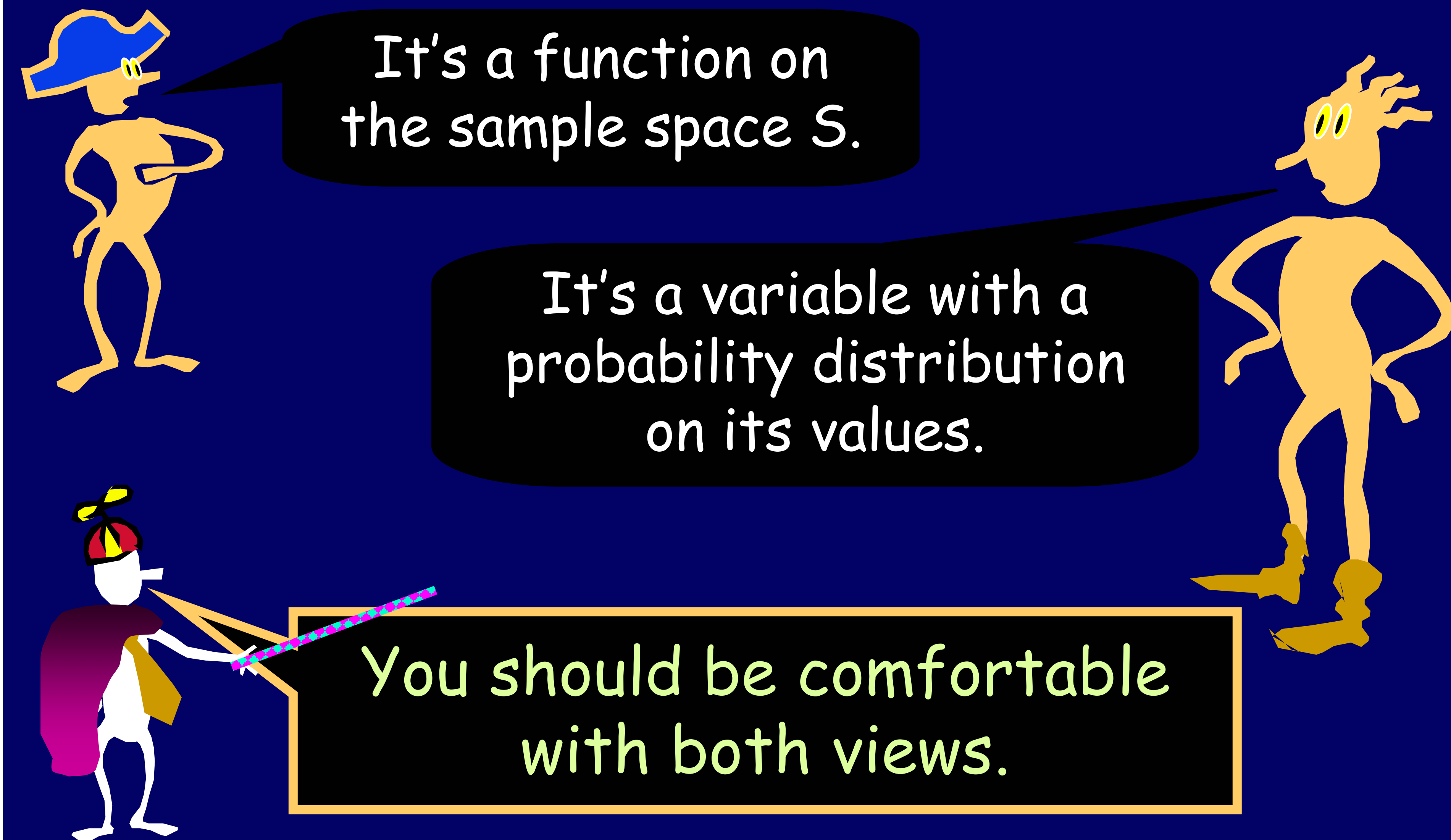


X = sum of both dice

function with X(1,1) = 2, X(1,2)=X(2,1)=3, …, X(6,6)=12

It's a floor wax *and* a dessert topping



From Random Variables to Events

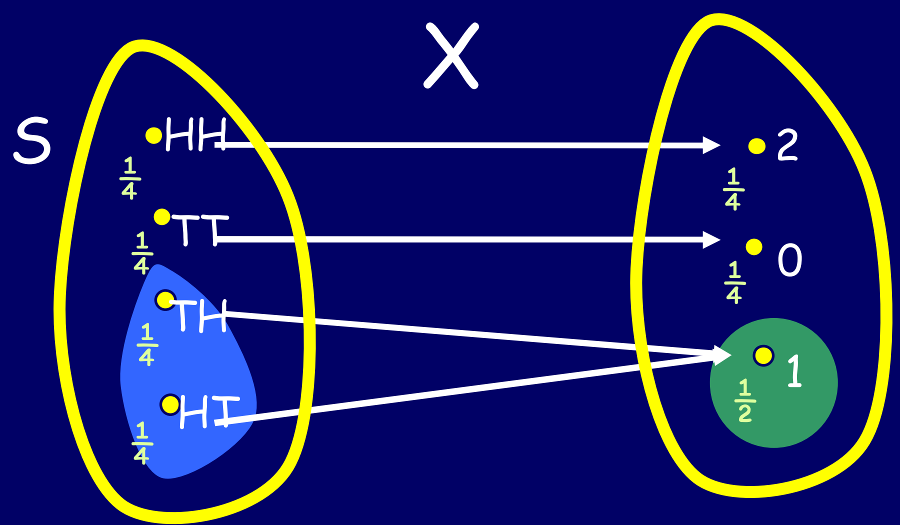
For any random variable X and value a,
we can define the event A that $X=a$.

$$\Pr(A) = \Pr(X=a) = \Pr(\{x \in S \mid X(x)=a\}).$$

Two coins tossed

X: {TT, TH, HT, HH} -> {0, 1, 2}
counts the number of heads

$$\Pr(X = a) = \Pr(\{x \in S \mid X(x) = a\})$$



Distribution on X

$\Pr(X = 1)$

$= \Pr(\{x \in S \mid X(x) = 1\})$

$= \Pr(\{TH, HT\}) = \frac{1}{2}.$

Two dice

I throw a white die and a black die. X = sum

Sample space S =

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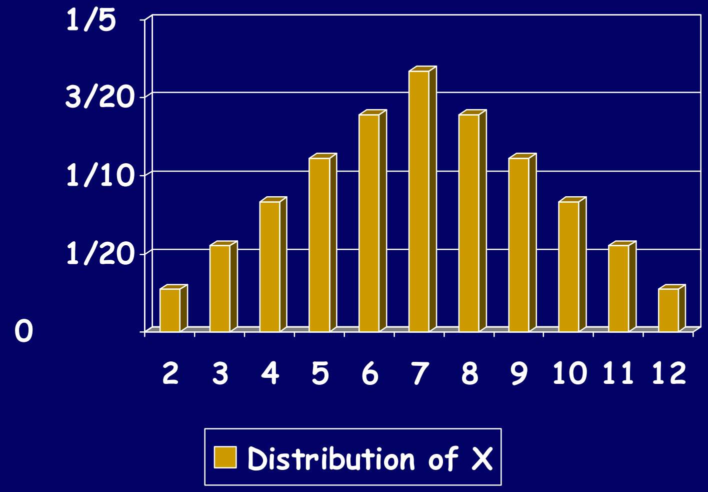


$\Pr(X = 6)$

$= \Pr(\{x \in S \mid X(x) = 6\})$

$= 5/36.$

$$\Pr(X = a) = \Pr(\{x \in S \mid X(x) = a\})$$

From Random Variables to Events

For any random variable X and value a, we can define the <u>event</u> A that $X=a$.

$$\Pr(A) = \Pr(X=a) = \Pr(\{x \in S \mid X(x)=a\}).$$

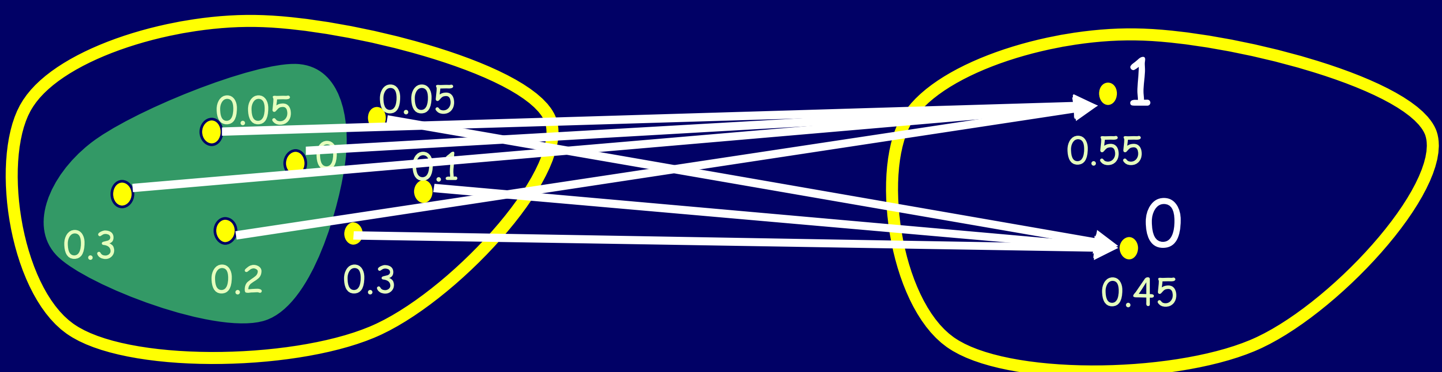
X has a distribution on its values

X is a function on the sample space S

From Events to Random Variables

For any event A, can define the __indicator random variable__ for A:

$$X_A(x) = 1 \quad \text{if } x \in A$$
$$= 0 \quad \text{if } x \notin A$$



Definition: expectation

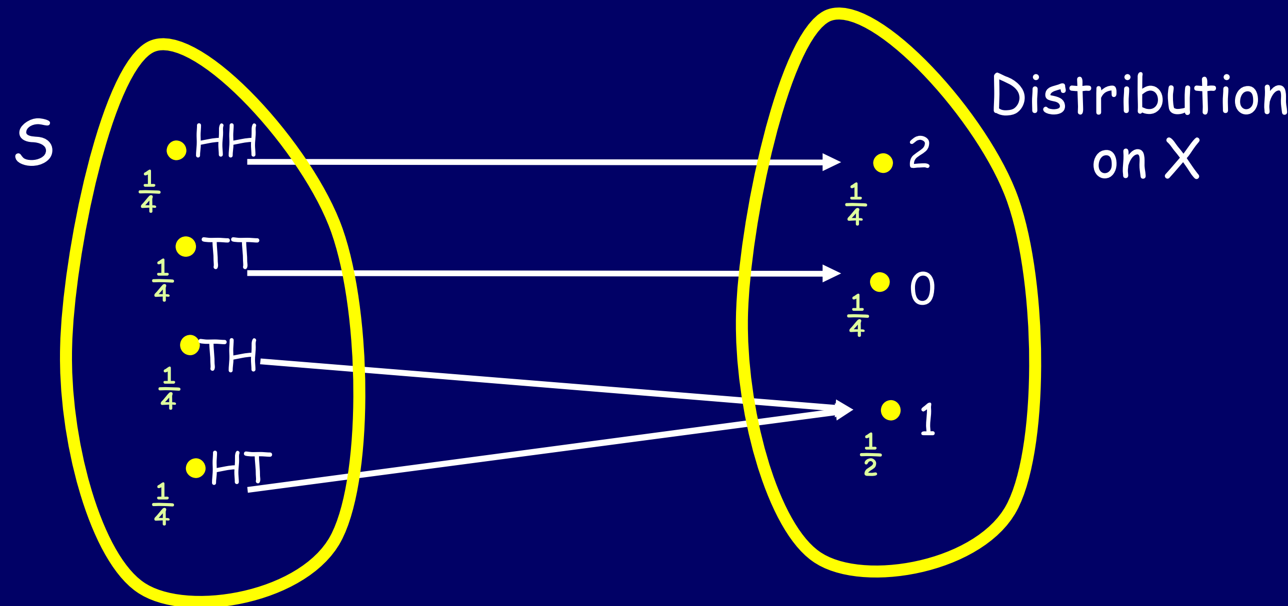
The expectation, or expected value of a random variable X is written as E[X], and is

$$E[X] = \sum_{x \in S} \Pr(x) X(x) = \sum_k k \Pr(X = k)$$

X is a function on the sample space S

X has a distribution on its values

Thinking about expectation



$$\boxed{\sum_{x \in S} X(x)\, Pr(x)} \qquad \boxed{\sum_k k\, Pr(X = k)}$$

$$E[X] = \tfrac{1}{4}*0 + \tfrac{1}{4}*1 + \tfrac{1}{4}*1 + \tfrac{1}{4}*2 = 1.$$

$$E[X] = \tfrac{1}{4}*0 + \tfrac{1}{2}*1 + \tfrac{1}{4}*2 = 1.$$

A quick calculation...

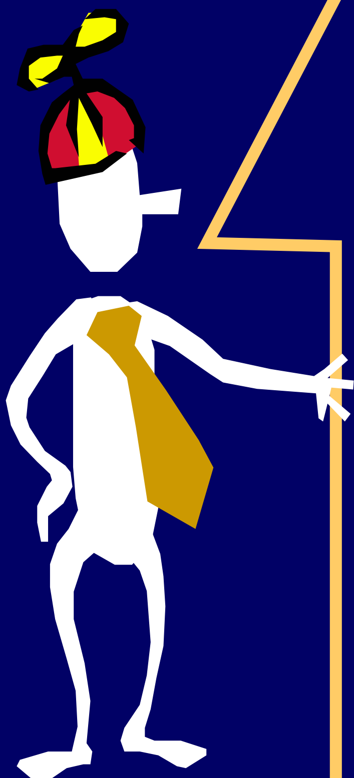
What if I flip a coin 3 times? Now what is the expected number of heads?

$E[X] = (1/8) \times 0 + (3/8) \times 1 + (3/8) \times 2 + (1/8) \times 3 = 1.5$

But $\Pr[\ X = 1.5\ ] = 0$...

Moral: don't always expect the expected.
$\Pr[\ X = E[X]\ ]$ may be 0 !

Type checking



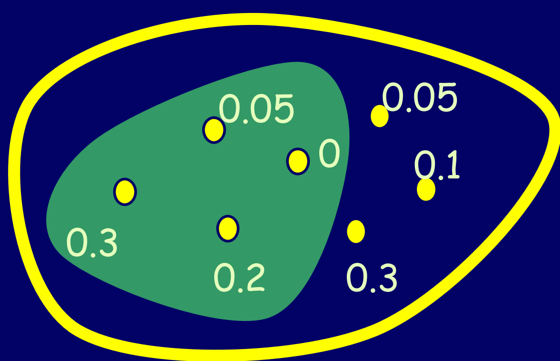
A Random Variable is the type of thing you might want to know an expected value of.

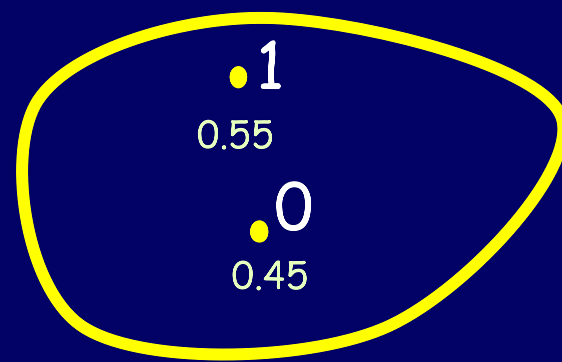
If you are computing an expectation, the thing whose expectation you are computing is a random variable.

Indicator R.V.s: $E[X_A] = Pr(A)$

For event A, the indicator random variable for A:

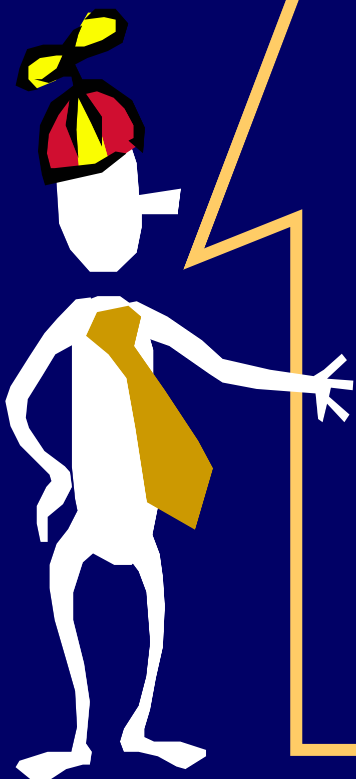
$$X_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$





$$E[X_A] = 1 \times Pr(X_A = 1) = Pr(A)$$

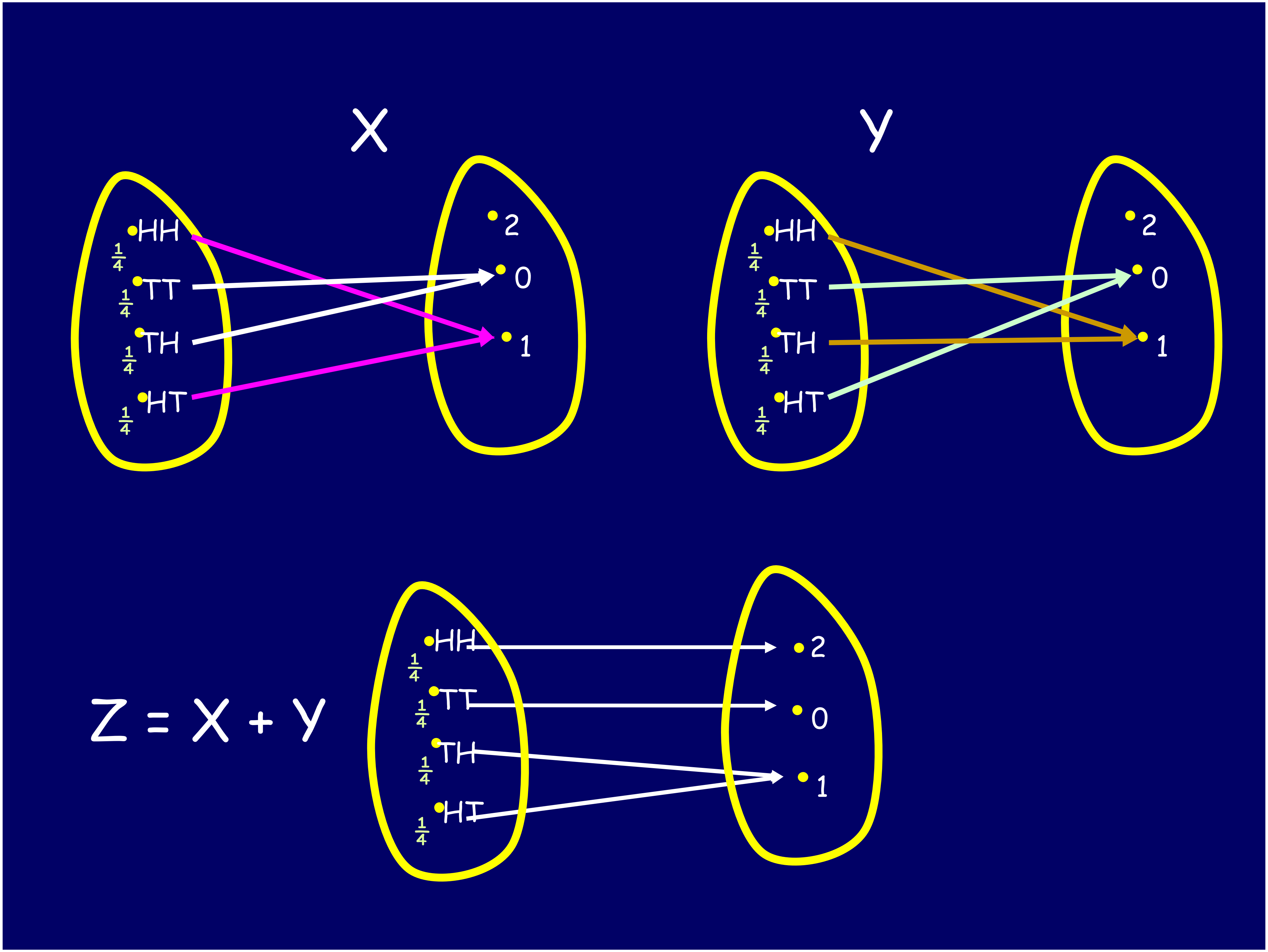
Adding Random Variables



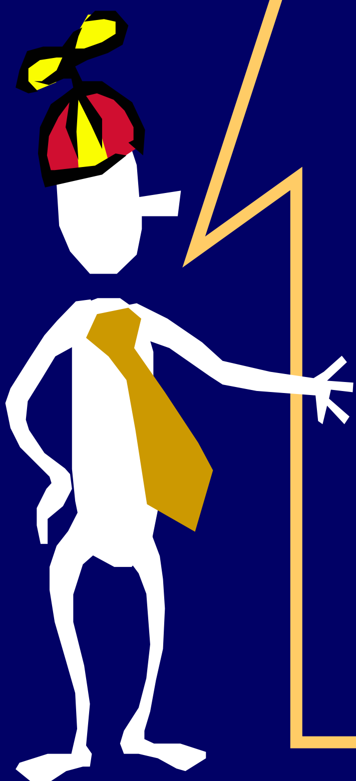
If X and Y are random variables (on the same set S), then $Z = X + Y$ is also a random variable.

$$Z(x) \equiv X(x) + Y(x)$$

E.g., rolling two dice.
X = 1st die, Y = 2nd die,
Z = sum of two dice.


X
Y
HH
TT
TH
HT
1/4
2
0
1
Z = X + Y

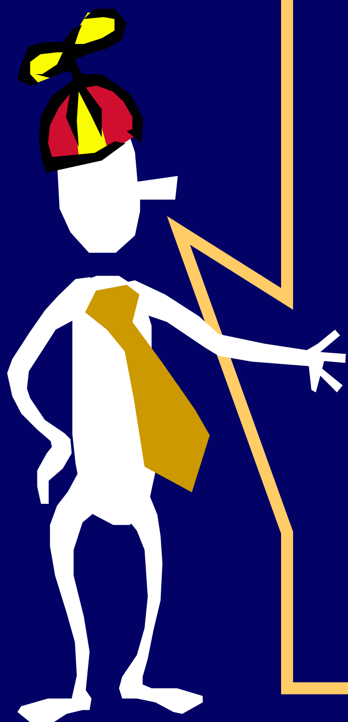
Adding Random Variables



Example: Consider picking a random person in the world. Let X = length of the person's left arm in inches. Y = length of the person's right arm in inches. Let $Z = X+Y$. Z measures the combined arm lengths.

Formally, S = {people in world}, D = uniform distribution on S.

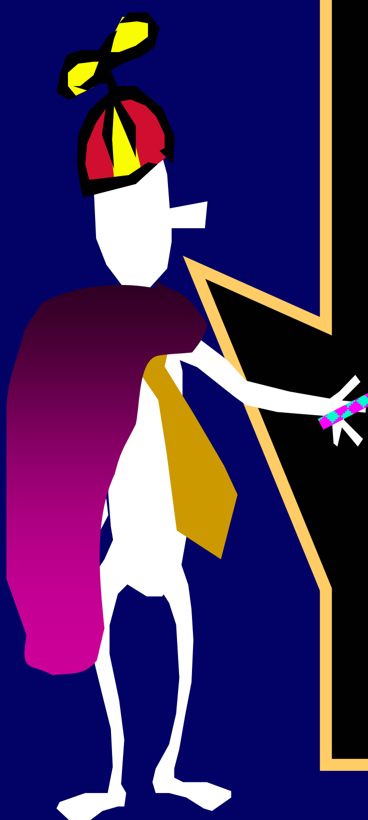
Independence



Two random variables X and Y are independent if for every a,b, the events X=a and Y=b are independent.

How about the case of
X=1st die, Y=2nd die?
X = left arm, Y=right arm?

Linearity of Expectation



If $Z = X+Y$, then

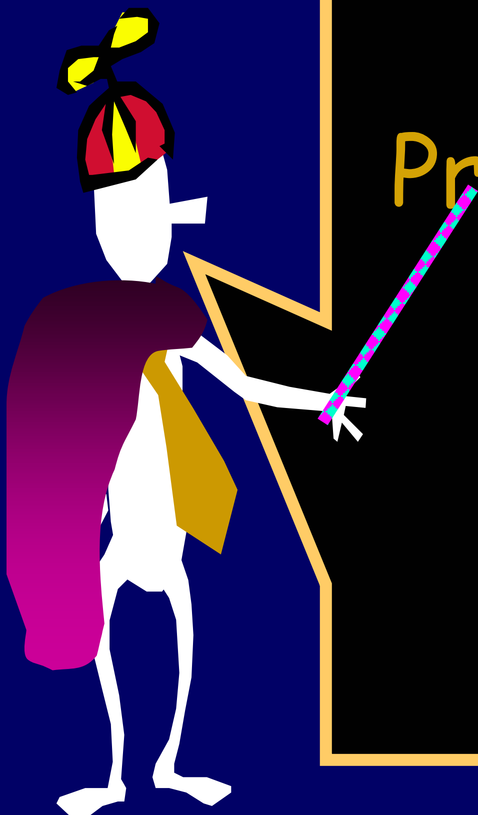
$$E[Z] = E[X] + E[Y]$$

Even if X and Y are not independent.

Linearity of Expectation

If $Z = X+Y$, then

$$E[Z] = E[X] + E[Y]$$



Proof:

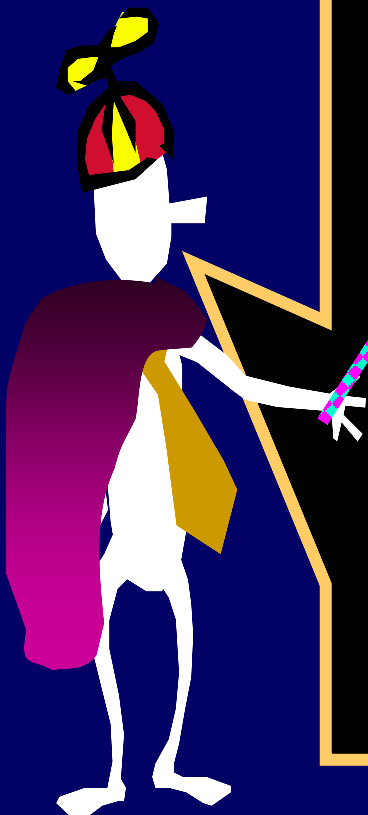
$$E[X] = \sum_{x \in S} \Pr(x)\, X(x)$$

$$E[Y] = \sum_{x \in S} \Pr(x)\, Y(x)$$

$$E[Z] = \sum_{x \in S} \Pr(x)\, Z(x)$$

but $Z(x) = X(x) + Y(x)$

Linearity of Expectation



E.g., 2 fair flips:

X = 1st coin, Y = 2nd coin.

$Z = X+Y$ = total # heads.

What is $E[X]$? $E[Y]$? $E[Z]$?

1,1,2
HH

1,0,1
HT

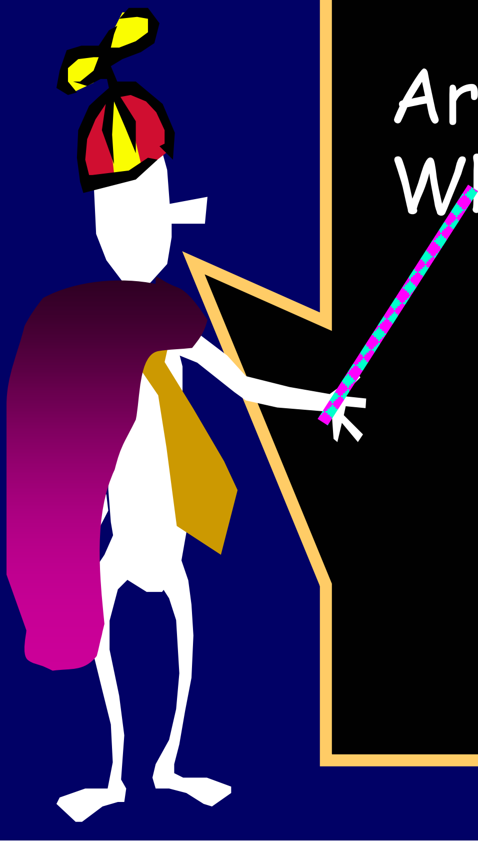
0,1,1
TH

0,0,0
TT

Linearity of Expectation

E.g., 2 fair flips:
X = at least one coin heads,
Y = both coins are heads, Z = X+Y

Are X and Y independent?
What is E[X] ? E[Y] ? E[Z] ?



1,1,2
HH

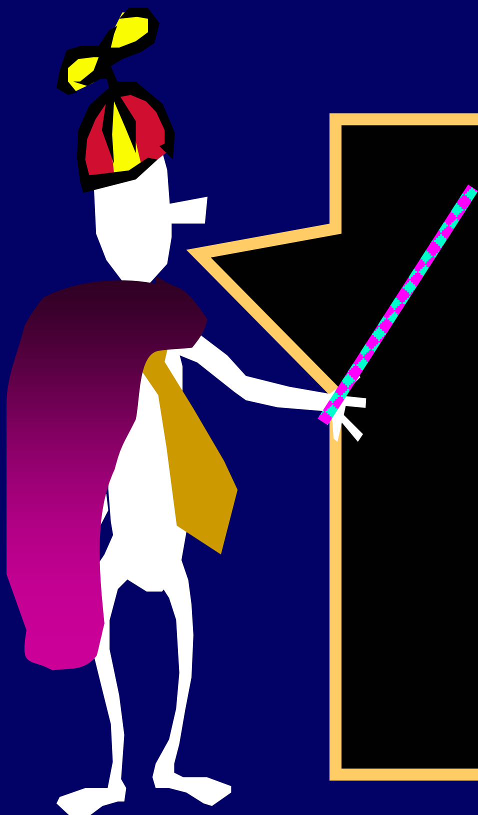
1,0,1
HT

1,0,1
TH

0,0,0
TT

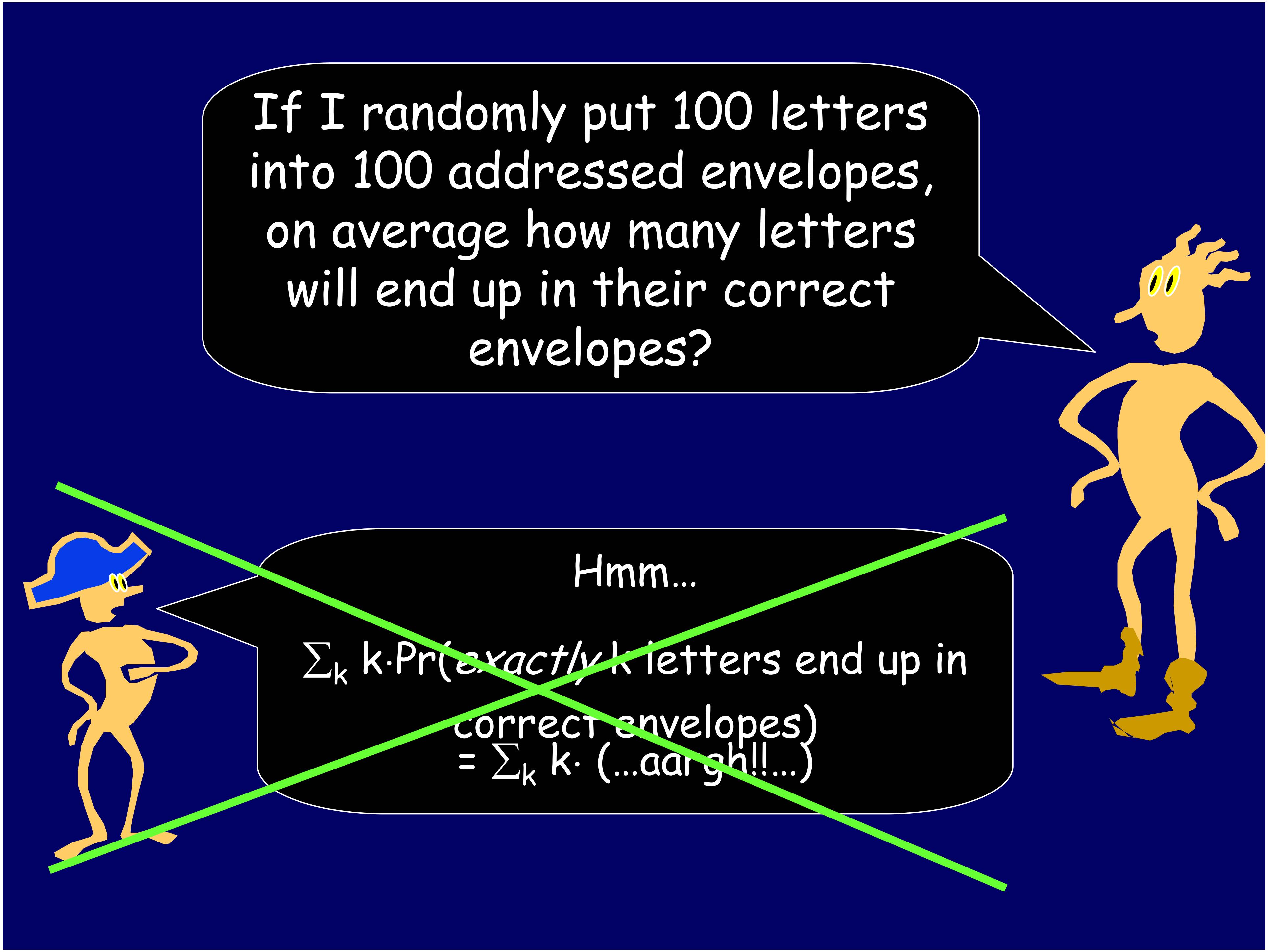
By induction

$$E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$$

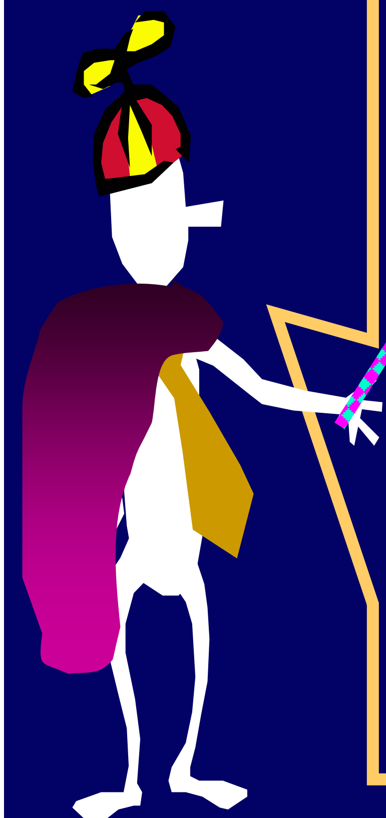


the expectation of the sum = the sum of the expectations


It is finally time to show off our probability prowess…


If I randomly put 100 letters into 100 addressed envelopes, on average how many letters will end up in their correct envelopes?
Hmm…
$\sum_k$ k·Pr(*exactly* k letters end up in correct envelopes)
= $\sum_k$ k· (…aargh!!…)

Use Linearity of Expectation



Let A_i be the event the i^{th} letter ends up in its correct envelope.

Let X_i be the indicator R.V. for A_i.

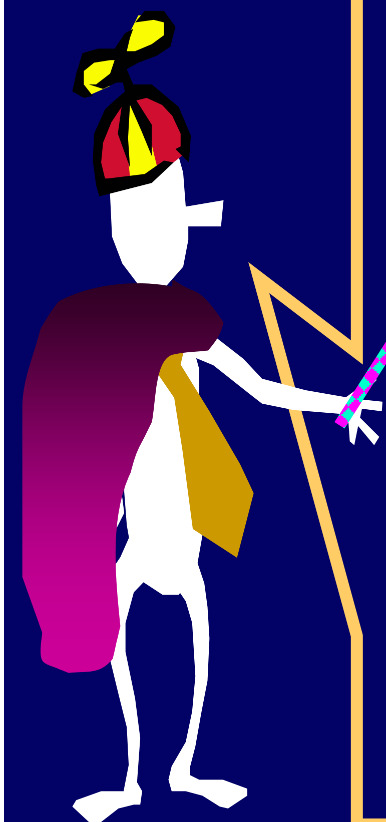
$$X_i = \begin{cases} 1 \text{ if } A_i \text{ occurs} \\ 0 \text{ otherwise} \end{cases}$$

Let $Z = X_1 + \ldots + X_{100}$.

We are asking for E[Z].

Use Linearity of Expectation

Let A_i be the event the i^{th} letter ends up in the correct envelope.
Let X_i be the indicator R.V. for A_i.
Let $Z = X_1 + \ldots + X_n$. We are asking for $E[Z]$.



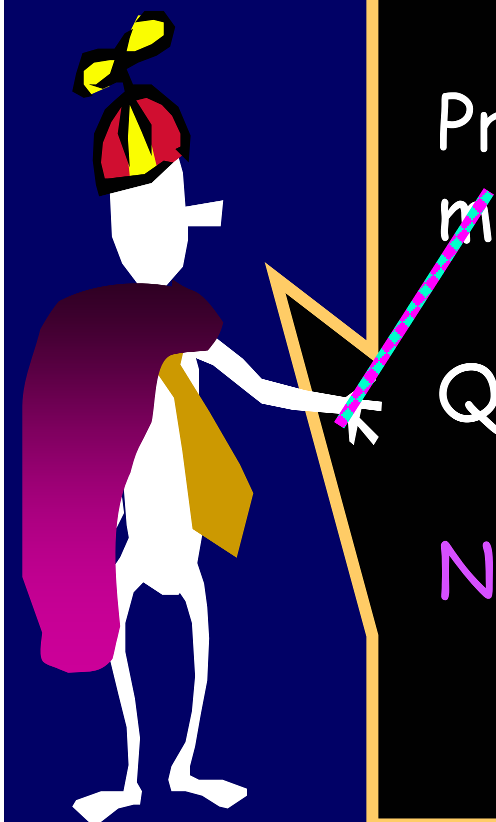
What is $E[X_i]$?

$$E[X_i] = \Pr(A_i) = 1/100.$$

What is $E[Z]$?

$$\begin{aligned} E[Z] &= E[X_1 + \ldots + X_{100}] \\ &= E[X_1] + \ldots + E[X_{100}] \\ &= 1/100 + \ldots + 1/100 = 1. \end{aligned}$$

Use Linearity of Expectation



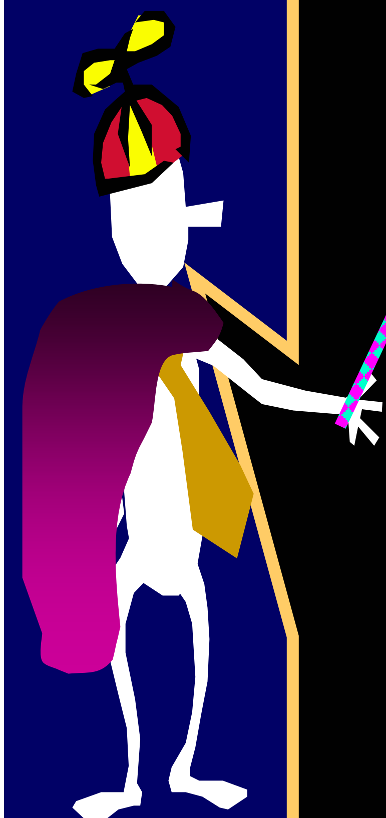
So, in expectation, 1 card will be in the same position as it started.

Pretty neat: it doesn't depend on how many cards!

Question: were the X_i independent?

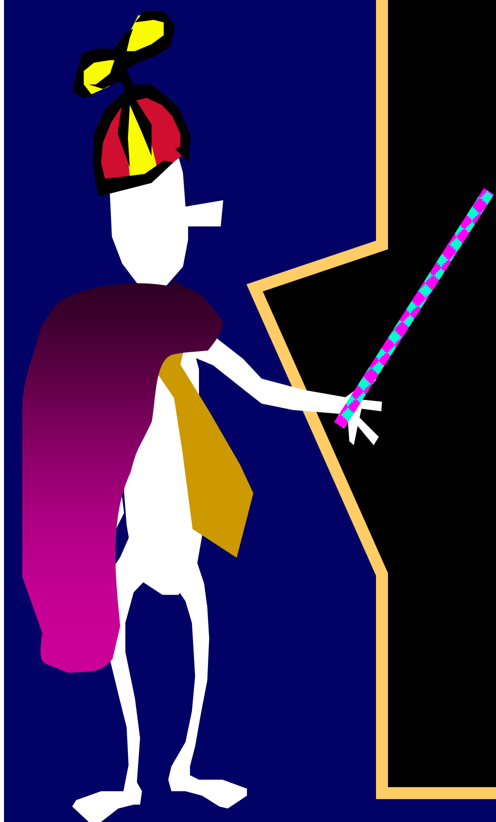
No! E.g., think of n=2.

Use Linearity of Expectation



General approach:

- View thing you care about as expected value of some RV.
- Write this RV as sum of simpler RVs (typically indicator RVs).
- Solve for their expectations and add them up!



Example

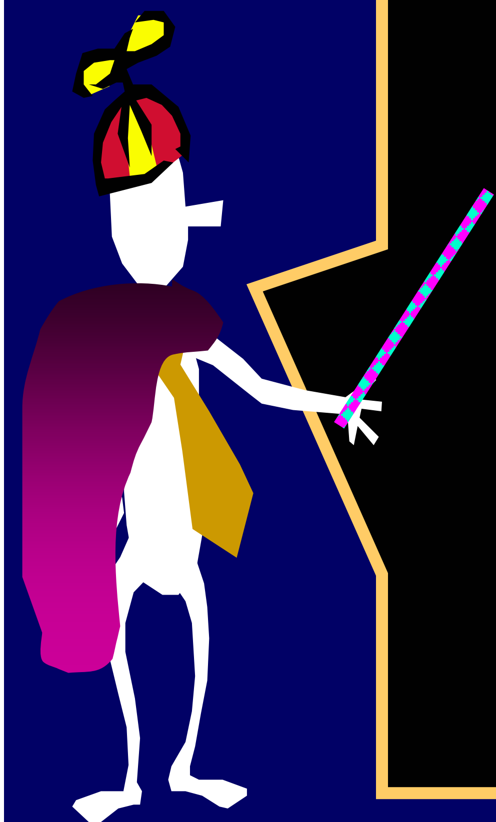
We flip n coins of bias p. What is the expected number of heads?

We could do this by summing

$$\sum_k k \Pr(X = k)$$

$$= \sum_k k \binom{n}{k} p^k (1-p)^{n-k}$$

But now we know a better way

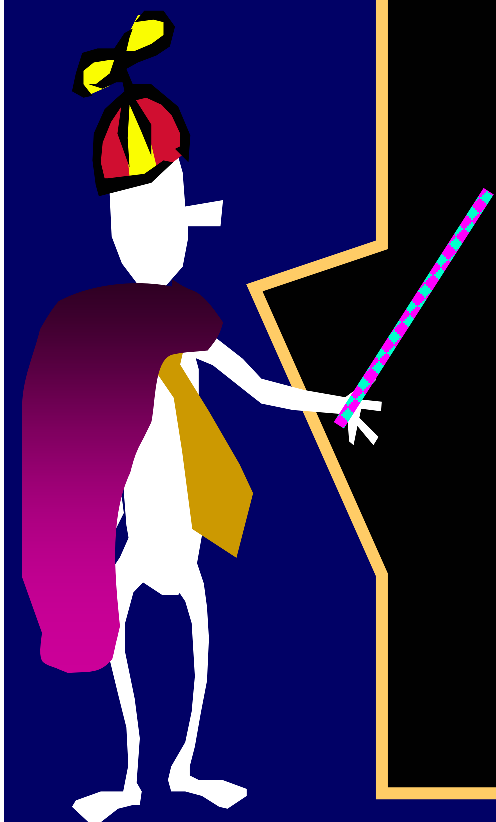


Let X = number of heads when n independent coins of bias p are flipped.

Break X into n simpler RVs,

$$X_i = \begin{cases} 0, & \text{if the } i^{th} \text{ coin is tails} \\ 1, & \text{if the } i^{th} \text{ coin is heads} \end{cases}$$

$$E[\ X\ ] = E[\ \Sigma_i\ X_i\ ] = ?$$



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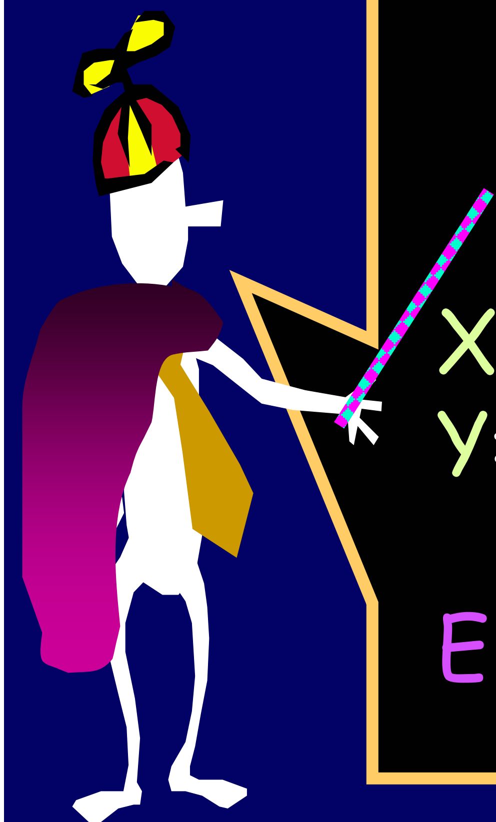
$$E[\,X\,] = E[\,\Sigma_i X_i\,] = np$$

What about Products?

If $Z = XY$, then

$E[Z] = E[X] \times E[Y]$?

No!

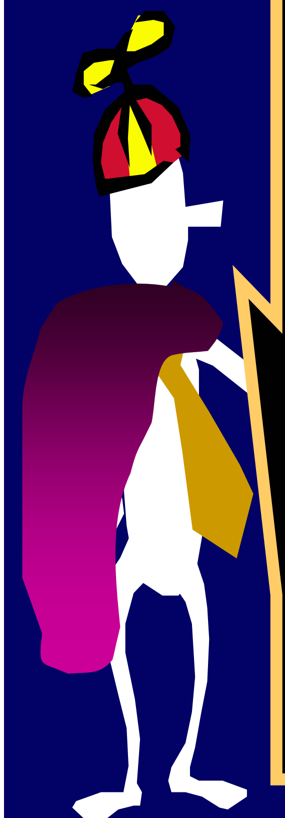


X=indicator for "1st flip is heads"

Y=indicator for "1st flip is tails".

$E[XY]=0$.

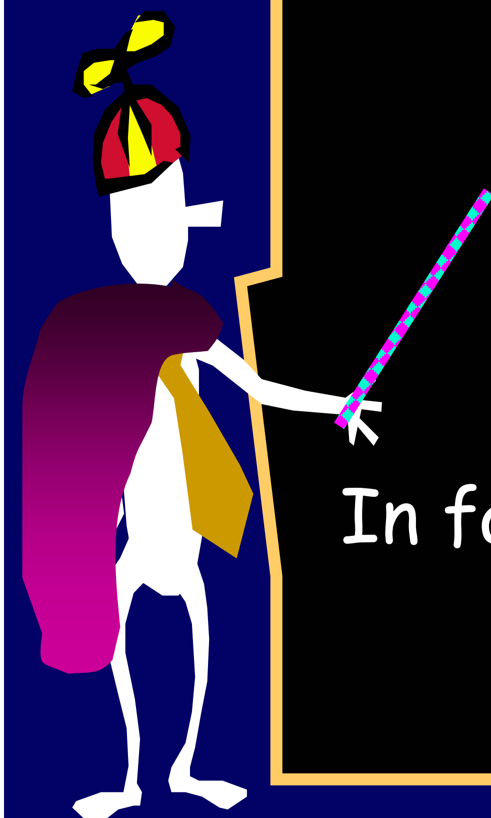
But it is true if RVs are independent



Proof:

$$E[X] = \textstyle\sum_a a \times \Pr(X=a)$$
$$E[Y] = \textstyle\sum_b b \times \Pr(Y=b)$$

$$\begin{aligned} E[XY] &= \textstyle\sum_c c \times \Pr(XY = c) \\ &= \textstyle\sum_c \sum_{a,b:ab=c} c \times \Pr(X=a \cap Y=b) \\ &= \textstyle\sum_{a,b} ab \times \Pr(X=a \cap Y=b) \\ &= \textstyle\sum_{a,b} ab \times \Pr(X=a) \Pr(Y=b) \\ &= E[X]\, E[Y] \end{aligned}$$



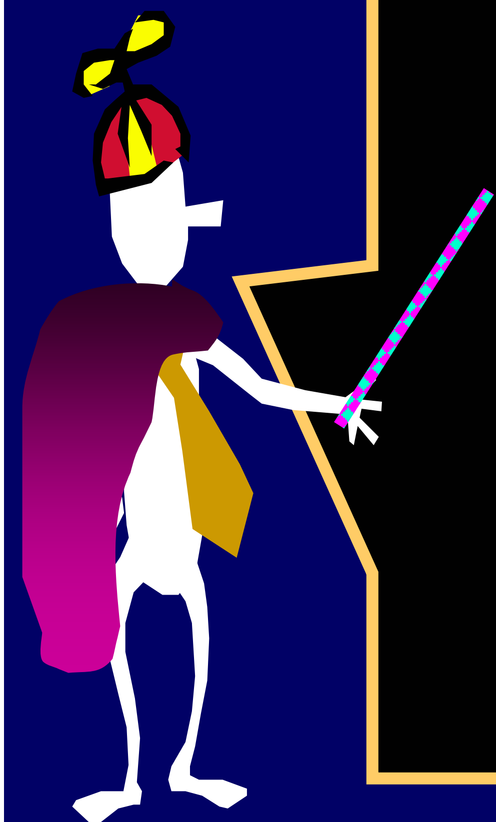
E.g., 2 fair flips.
X = indicator for 1st coin being heads,
Y = indicator for 2nd coin being heads.
XY = indicator for "both are heads".

$$E[X] = \tfrac{1}{2},\ E[Y] = \tfrac{1}{2},\ E[XY] = \tfrac{1}{4}.$$

$$E[X*X] = E[X]^2?$$

No: $E[X^2] = \tfrac{1}{2}$, $E[X]^2 = \tfrac{1}{4}$.

In fact, $E[X^2] - E[X]^2$ is called the *variance* of X.



Most of the time, though, power will come from using sums.

Mostly because Linearity of Expectations holds even if RVs are <u>not independent</u>.

Another problem

On average, in class of size m, how many pairs of people will have the same birthday?

~~$\sum_k k\cdot$ Pr(exactly k collisions)~~

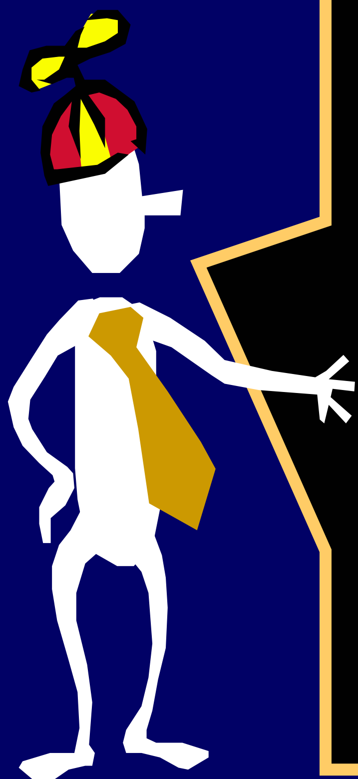
~~$= \sum_k k\cdot$ (...aargh!!!!...)~~

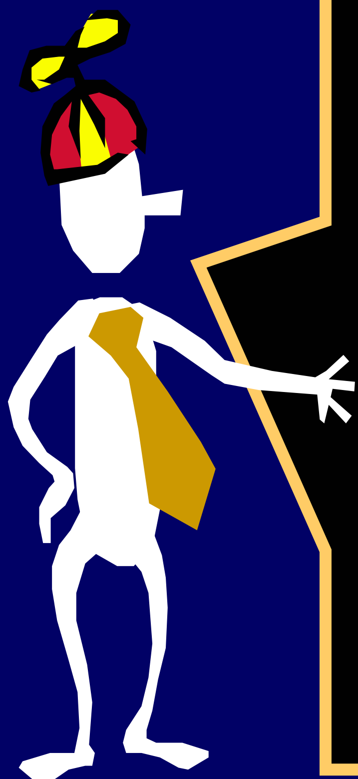
Use linearity of expectation.

Suppose we have m people each with a uniformly chosen birthday from 1 to 366.

X = number of pairs of people with the same birthday.

E[X] = ?





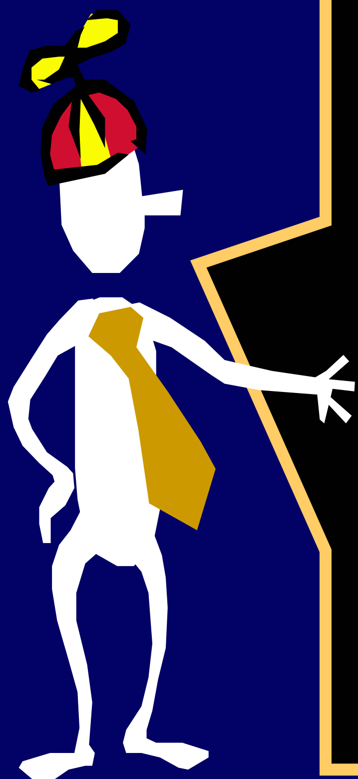
X = number of pairs of people with the same birthday.

$E[X] = ?$

Use $m(m-1)/2$ indicator variables, one for each pair of people.

$X_{jk} = 1$ if person j and person k have the same birthday; else 0.

$$E[X_{jk}] = (1/366)\,1 + (1 - 1/366)\,0$$
$$= 1/366$$

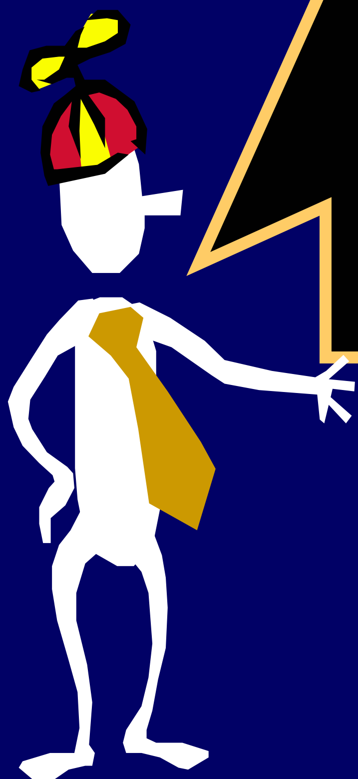


X = number of pairs of people with the same birthday.

$$E[X] = E[\ \Sigma_{j \leq k \leq m} X_{jk}\ ]$$

There are many dependencies among the indicator variables. E.g., X_{12} and X_{13} and X_{23} are dependent.

But we don't care!



X = number of pairs of people with the same birthday.

$$E[X] = E[\ \Sigma_{j \leq k \leq m} X_{jk}\ ]$$

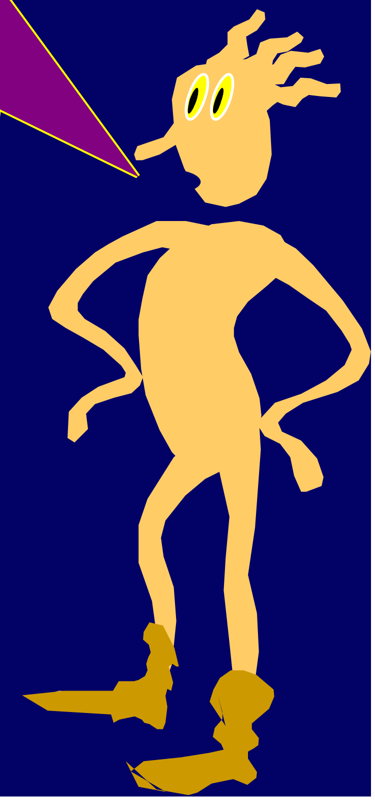
$$= \Sigma_{j \leq k \leq m} E[\ X_{jk}\ ]$$

$$= m(m-1)/2 \times 1/366$$

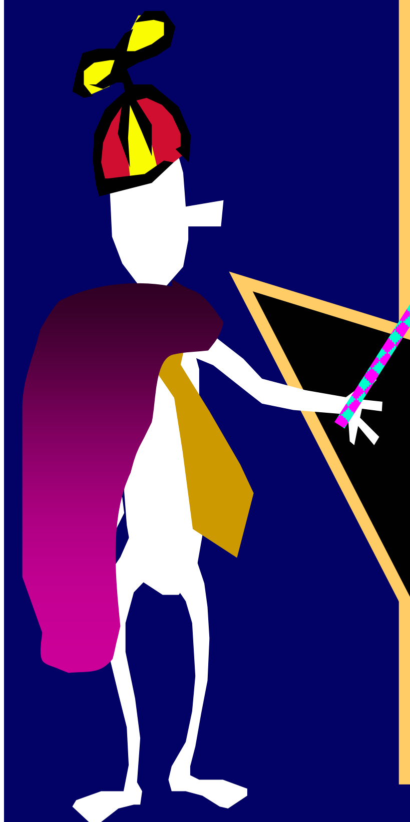
Step right up…

You pick a number $n \in [1..6]$. You roll 3 dice. If any match n, you win \$1. Else you pay me \$1. Want to play?

Hmm… let's see



Analysis



A_i = event that i-th die matches.

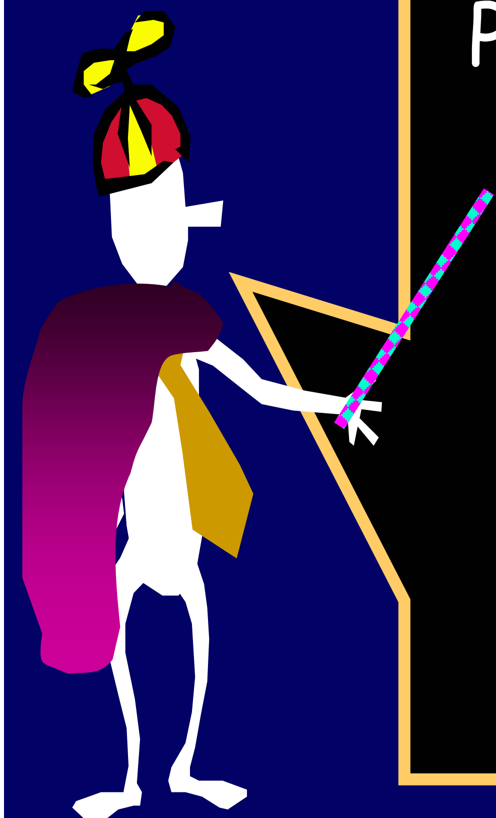
X_i = indicator RV for A_i.

Expected number of dice that match:

$$E[X_1+X_2+X_3] = 1/6+1/6+1/6 = \tfrac{1}{2}.$$

But this is not the same as Pr(at least one die matches).

Analysis



Pr(at least one die matches)

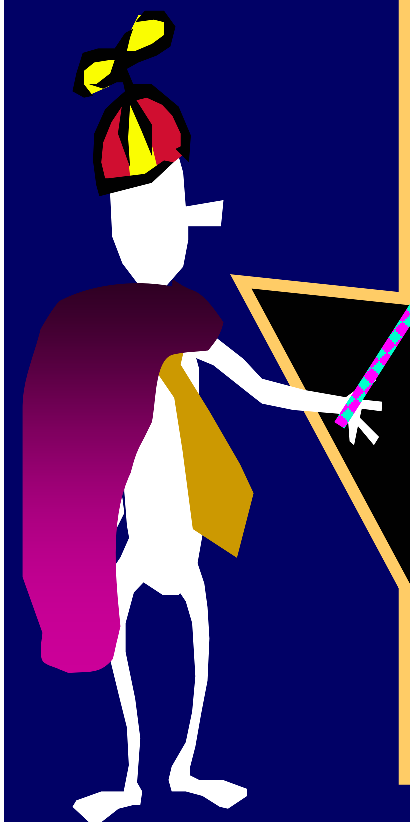
$= 1 - \Pr(\text{none match})$

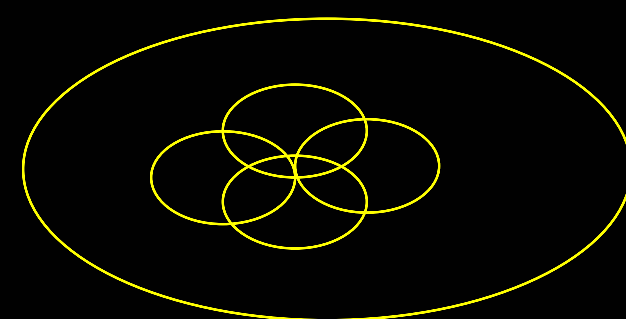
$= 1 - (5/6)^3 = 0.416.$

What's going on?

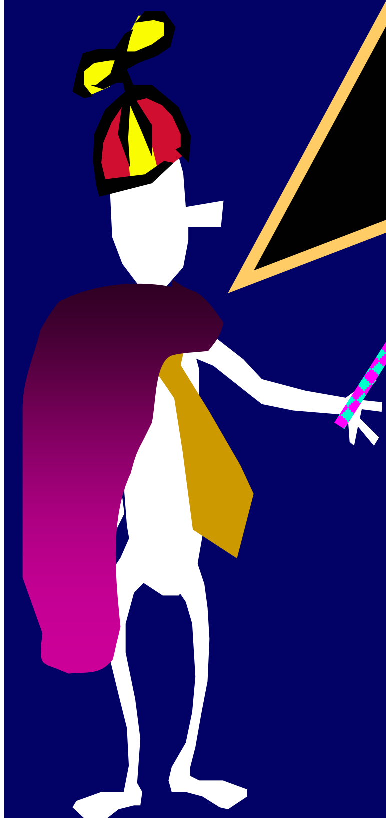
Say we have a collection of events $A_1, A_2, \ldots$.

How does E[# events that occur] compare to Pr(at least one occurs)?





What's going on?



E[# events that occur]

$= \sum_k \Pr(k \text{ events occur}) \times k$

$= \sum_{(k > 0)} \Pr(k \text{ events occur}) \times k$

Pr(at least one event occurs)

$= \sum_{(k > 0)} \Pr(k \text{ events occur})$

What's going on?

Moral #1: be careful you are modeling problem correctly.

Moral #2: watch out for carnival games.